# Analysis

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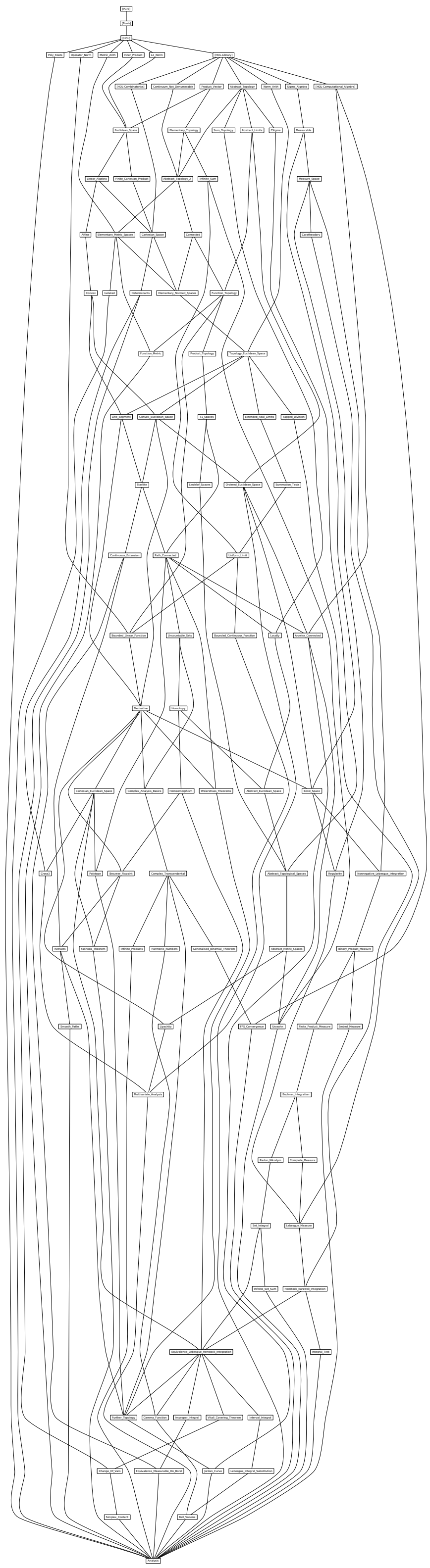

# Chapter 1

# Linear Algebra

**theory** *L2_Norm*
**imports** *Complex_Main*
**begin**

## 1.1 L2 Norm

**definition** *L2_set* :: $('a \Rightarrow real) \Rightarrow 'a\ set \Rightarrow real$ **where**
*L2_set f A* = *sqrt* $(\sum i{\in}A.\ (f\ i)^2)$

**proposition** *L2_set_triangle_ineq*:
*L2_set* $(\lambda i.\ f\ i + g\ i)$ $A \leq$ *L2_set f A* + *L2_set g A*

**end**

## 1.2 Inner Product Spaces and Gradient Derivative

**theory** *Inner_Product*
**imports** *Complex_Main*
**begin**

### 1.2.1 Real inner product spaces

**class** *real_inner* = *real_vector* + *sgn_div_norm* + *dist_norm* + *uniformity_dist* + *open_uniformity* +
**fixes** *inner* :: $'a \Rightarrow 'a \Rightarrow real$
**assumes** *inner_commute*: *inner x y* = *inner y x*
**and** *inner_add_left*: *inner* $(x + y)$ *z* = *inner x z* + *inner y z*
**and** *inner_scaleR_left* [*simp*]: *inner* (*scaleR r x*) *y* = $r *$ (*inner x y*)
**and** *inner_ge_zero* [*simp*]: $0 \leq$ *inner x x*
**and** *inner_eq_zero_iff* [*simp*]: *inner x x* $= 0 \longleftrightarrow x = 0$
**and** *norm_eq_sqrt_inner*: *norm x* = *sqrt* (*inner x x*)
**begin**

### 1.2.2 Class instances

**instantiation** *real* :: *real_inner*
**begin**

**instantiation** *complex* :: *real_inner*
**begin**

### 1.2.3 Gradient derivative

**definition**
*gderiv* ::
['a::real_inner ⇒ real, 'a, 'a] ⇒ bool
((GDERIV (_)/ (_)/ :> (_)) [1000, 1000, 60] 60)
**where**
*GDERIV f x* :> *D* ⟷ *FDERIV f x* :> (λh. *inner h D*)

**end**

## 1.3 Cartesian Products as Vector Spaces

**theory** *Product_Vector*
**imports**
*Complex_Main*
*HOL−Library.Product_Plus*
**begin**

### 1.3.1 Product is a Module

**lemma** *scale_prod*: *scale x* (*a*, *b*) = (*s1 x a*, *s2 x b*)

**sublocale** *p*: *module scale*

### 1.3.2 Product is a Real Vector Space

**instantiation** *prod* :: (*real_vector*, *real_vector*) *real_vector*
**begin**

**proposition** *scaleR_Pair* [*simp*]: *scaleR r* (*a*, *b*) = (*scaleR r a*, *scaleR r b*)

### 1.3.3 Product is a Metric Space

**class** *uniform_topological_monoid_add* = *topological_monoid_add* + *uniform_space*
+
**assumes** *uniformly_continuous_add'*:
*filterlim* (λ((*a*,*b*), (*c*,*d*)). (*a* + *c*, *b* + *d*)) *uniformity* (*uniformity* $\times_F$ *uniformity*)

**class** *uniform_topological_group_add* = *topological_group_add* + *uniform_topological_monoid_add*
+
**assumes** *uniformly_continuous_uminus'*: *filterlim* (λ(*a*, *b*). (−*a*, −*b*)) *uniformity*
*uniformity*
**begin**

**instantiation** *prod* :: (*metric_space*, *metric_space*) *metric_space*
**begin**

**proposition** *dist_Pair_Pair*: *dist* (*a*, *b*) (*c*, *d*) = *sqrt* ((*dist a c*)$^2$ + (*dist b d*)$^2$)

### 1.3.4 Product is a Complete Metric Space

**instance** *prod* :: (*complete_space*, *complete_space*) *complete_space*

### 1.3.5 Product is a Normed Vector Space

**instantiation** *prod* :: (*real_normed_vector*, *real_normed_vector*) *real_normed_vector*
**begin**

**proposition** *norm_Pair*: *norm* (*a*, *b*) = *sqrt* ((*norm a*)$^2$ + (*norm b*)$^2$)

**instance** *prod* :: (*banach*, *banach*) *banach*

**proposition** *has_derivative_Pair* [*derivative_intros*]:
**assumes** *f*: (*f has_derivative f'*) (*at x within s*)
**and** *g*: (*g has_derivative g'*) (*at x within s*)
**shows** ((λ*x*. (*f x*, *g x*)) *has_derivative* (λ*h*. (*f' h*, *g' h*))) (*at x within s*)

### 1.3.6 Product is Finite Dimensional

**proposition** *dim_Times*:
**assumes** *vs1.subspace S vs2.subspace T*
**shows** *p.dim*(*S* × *T*) = *vs1.dim S* + *vs2.dim T*

**end**

## 1.4 Finite-Dimensional Inner Product Spaces

**theory** *Euclidean_Space*
**imports**
*L2_Norm*
*Inner_Product*
*Product_Vector*
**begin**

### 1.4.1 Type class of Euclidean spaces

**class** *euclidean_space* = *real_inner* +
**fixes** *Basis* :: *'a set*
**assumes** *nonempty_Basis* [*simp*]: *Basis* ≠ {}
**assumes** *finite_Basis* [*simp*]: *finite Basis*
**assumes** *inner_Basis*:
⟦*u* ∈ *Basis*; *v* ∈ *Basis*⟧ ⟹ *inner u v* = (*if u* = *v then 1 else 0*)
**assumes** *euclidean_all_zero_iff*:
(∀ *u*∈*Basis*. *inner x u* = *0*) ⟷ (*x* = *0*)

### 1.4.2 Class instances

**instantiation** *real* :: *euclidean_space*
**begin**
**instantiation** *complex* :: *euclidean_space*
**begin**
**instantiation** *prod* :: (*real_inner*, *real_inner*) *real_inner*
**begin**

**instantiation** *prod* :: (*euclidean_space*, *euclidean_space*) *euclidean_space*
**begin**

### 1.4.3 Locale instances

**end**

## 1.5 Elementary Linear Algebra on Euclidean Spaces

**theory** *Linear_Algebra*
**imports**
*Euclidean_Space*
*HOL−Library.Infinite_Set*
**begin**

### 1.5.1 Substandard Basis

### 1.5.2 Orthogonality

**definition** (**in** *real_inner*) *orthogonal x y* $\longleftrightarrow$ *x* $\cdot$ *y* = *0*

### 1.5.3 Orthogonality of a transformation

**definition** *orthogonal_transformation f* $\longleftrightarrow$ *linear f* $\wedge$ ($\forall$ *v w*. *f v* $\cdot$ *f w* = *v* $\cdot$ *w*)

### 1.5.4 Bilinear functions

**definition**
*bilinear* :: (*'a*::*real_vector* $\Rightarrow$ *'b*::*real_vector* $\Rightarrow$ *'c*::*real_vector*) $\Rightarrow$ *bool* **where**
*bilinear f* $\longleftrightarrow$ ($\forall$ *x*. *linear* ($\lambda$*y*. *f x y*)) $\wedge$ ($\forall$ *y*. *linear* ($\lambda$*x*. *f x y*))

### 1.5.5 Adjoints

**definition** *adjoint* :: ((*'a*::*real_inner*) $\Rightarrow$ (*'b*::*real_inner*)) $\Rightarrow$ *'b* $\Rightarrow$ *'a* **where**
*adjoint f* = (*SOME f'*. $\forall$ *x y*. *f x* $\cdot$ *y* = *x* $\cdot$ *f' y*)

### 1.5.6 Infinity norm

**definition** *infnorm* (*x*::*'a*::*euclidean_space*) = *Sup* {|*x* $\cdot$ *b*| |*b*. *b* $\in$ *Basis*}

### 1.5.7 Collinearity

**definition** *collinear* :: *'a*::*real_vector set* $\Rightarrow$ *bool*
 **where** *collinear S* $\longleftrightarrow$ ($\exists$ *u*. $\forall$ *x* $\in$ *S*. $\forall$ *y* $\in$ *S*. $\exists$ *c*. *x* − *y* = *c* $*_R$ *u*)

### 1.5.8 Properties of special hyperplanes

**proposition** *dim_hyperplane*:
 **fixes** *a* :: *'a*::*euclidean_space*
 **assumes** *a* $\neq$ *0*
  **shows** *dim* {*x*. *a* $\cdot$ *x* = *0*} = *DIM*(*'a*) − *1*

### 1.5.9 Orthogonal bases and Gram-Schmidt process

**proposition** *Gram_Schmidt_step*:
 **fixes** *S* :: *'a*::*euclidean_space set*
 **assumes** *S*: *pairwise orthogonal S* **and** *x*: *x* $\in$ *span S*

**shows** *orthogonal x* $(a - (\sum b{\in}S.\ (b \cdot a\ /\ (b \cdot b)) *_R\ b))$

**proposition** *orthogonal_extension*:
**fixes** $S$ :: *'a::euclidean_space set*
**assumes** $S$: *pairwise orthogonal S*
**obtains** $U$ **where** *pairwise orthogonal* $(S \cup U)$ *span* $(S \cup U) =$ *span* $(S \cup T)$

### 1.5.10 Decomposing a vector into parts in orthogonal subspaces

**proposition** *orthonormal_basis_subspace*:
**fixes** $S$ :: *'a :: euclidean_space set*
**assumes** *subspace S*
**obtains** $B$ **where** $B \subseteq S$ *pairwise orthogonal B*
**and** $\bigwedge x.\ x \in B \Longrightarrow$ *norm* $x = 1$
**and** *independent B card B* $=$ *dim S span B* $= S$

**proposition** *dim_orthogonal_sum*:
**fixes** $A$ :: *'a::euclidean_space set*
**assumes** $\bigwedge x\ y.\ [\![x \in A;\ y \in B]\!] \Longrightarrow x \cdot y = 0$
**shows** *dim*$(A \cup B) =$ *dim* $A$ $+$ *dim* $B$

### 1.5.11 Linear functions are (uniformly) continuous on any set

**end**

## 1.6 Affine Sets

**theory** *Affine*
**imports** *Linear_Algebra*
**begin**

### 1.6.1 Affine set and affine hull

**definition** *affine* :: *'a::real_vector set* $\Rightarrow$ *bool*
**where** *affine s* $\longleftrightarrow (\forall x{\in}s.\ \forall y{\in}s.\ \forall u\ v.\ u + v = 1 \longrightarrow u *_R x + v *_R y \in s)$

### 1.6.2 Affine Dependence

**definition** *affine_dependent* :: *'a::real_vector set* ⇒ *bool*
**where** $affine\_dependent\ s \longleftrightarrow (\exists x{\in}s.\ x \in affine\ hull\ (s - \{x\}))$

**proposition** *affine_dependent_explicit*:
$affine\_dependent\ p \longleftrightarrow$
$(\exists S\ u.\ finite\ S \wedge S \subseteq p \wedge sum\ u\ S = 0 \wedge (\exists v{\in}S.\ u\ v \neq 0) \wedge sum\ (\lambda v.\ u\ v *_R v)\ S = 0)$

**proposition** *extend_to_affine_basis*:
**fixes** *S V* :: *'n::real_vector set*
**assumes** $\neg$ *affine_dependent S S* $\subseteq$ *V*
**obtains** *T* **where** $\neg$ *affine_dependent T S* $\subseteq$ *T T* $\subseteq$ *V affine hull T* = *affine hull V*

### 1.6.3 Affine Dimension of a Set

**definition** *aff_dim* :: *('a::euclidean_space) set* ⇒ *int*
**where** *aff_dim V* =
(*SOME d* :: *int*.
$\exists B.\ affine\ hull\ B = affine\ hull\ V \wedge \neg\ affine\_dependent\ B \wedge of\_nat\ (card\ B) = d + 1)$

**end**

## 1.7 Convex Sets and Functions

**theory** *Convex*
**imports**
*Affine*
*HOL−Library.Set_Algebras*
**begin**

### 1.7.1 Convex Sets

**definition** *convex* :: *'a::real_vector set* ⇒ *bool*
**where** $convex\ s \longleftrightarrow (\forall x{\in}s.\ \forall y{\in}s.\ \forall u{\geq}0.\ \forall v{\geq}0.\ u + v = 1 \longrightarrow u *_R x + v *_R y \in s)$

### 1.7.2 Convex Functions on a Set

**definition** *convex_on* :: *'a::real_vector set* ⇒ *('a ⇒ real)* ⇒ *bool*
**where** $convex\_on\ S\ f \longleftrightarrow$
$(\forall x{\in}S.\ \forall y{\in}S.\ \forall u{\geq}0.\ \forall v{\geq}0.\ u + v = 1 \longrightarrow f\ (u *_R x + v *_R y) \leq u * f\ x + v * f\ y)$

**definition** *concave_on* :: $'a$::*real_vector set* $\Rightarrow$ ($'a \Rightarrow$ *real*) $\Rightarrow$ *bool*
  **where** *concave_on S f* $\equiv$ *convex_on S* ($\lambda x. - f\ x$)

### 1.7.3 Some inequalities

### 1.7.4 Misc related lemmas

### 1.7.5 Cones

**definition** *cone* :: $'a$::*real_vector set* $\Rightarrow$ *bool*
  **where** *cone s* $\longleftrightarrow$ ($\forall x \in s.\ \forall c \geq 0.\ c *_R x \in s$)

**proposition** *cone_hull_expl*: *cone hull S* $= \{c *_R x \mid c\ x.\ c \geq 0 \wedge x \in S\}$
  (**is** *?lhs* = *?rhs*)

### 1.7.6 Convex hull

**proposition** *convex_hull_indexed*:
  **fixes** $S$ :: $'a$::*real_vector set*
  **shows** *convex hull S* =
    $\{y.\ \exists k\ u\ x.\ (\forall i \in \{1::nat\ ..\ k\}.\ 0 \leq u\ i \wedge x\ i \in S) \wedge$
      $(sum\ u\ \{1..k\} = 1) \wedge (\sum i = 1..k.\ u\ i *_R x\ i) = y\}$
    (**is** *?xyz* = *?hull*)

### 1.7.7 Caratheodory's theorem

**theorem** *caratheodory*:
  *convex hull p* =
    $\{x::'a::euclidean\_space.\ \exists S.\ finite\ S \wedge S \subseteq p \wedge card\ S \leq DIM('a) + 1 \wedge x \in$ *convex hull S*$\}$

### 1.7.8 Radon's theorem

**theorem** *Radon*:
  **assumes** *affine_dependent c*
  **obtains** *M P* **where** $M \subseteq c$ $P \subseteq c$ $M \cap P = \{\}$ (*convex hull M*) $\cap$ (*convex hull P*) $\neq \{\}$

### 1.7.9 Helly's theorem

**theorem** *Helly*:
  **fixes** $\mathcal{F}$ :: $'a$::*euclidean_space set set*

**assumes** $card\ \mathcal{F} \geq DIM('a) + 1\ \forall s{\in}\mathcal{F}.\ convex\ s$
**and** $\bigwedge t.\ [\![t{\subseteq}\mathcal{F};\ card\ t = DIM('a) + 1]\!] \Longrightarrow \bigcap t \neq \{\}$
**shows** $\bigcap \mathcal{F} \neq \{\}$

### 1.7.10 Epigraphs of convex functions

**definition** $epigraph\ S\ (f :: \_ \Rightarrow real) = \{xy.\ fst\ xy \in S \wedge f\ (fst\ xy) \leq snd\ xy\}$

**end**

## 1.8 Definition of Finite Cartesian Product Type

**theory** *Finite_Cartesian_Product*
**imports**
*Euclidean_Space*
*L2_Norm*
*HOL−Library.Numeral_Type*
*HOL−Library.Countable_Set*
*HOL−Library.FuncSet*
**begin**

### 1.8.1 Cardinality of vectors

**proposition** *CARD_vec* [*simp*]:
$CARD('a\hat{\ }'b) = CARD('a)\ \hat{}\ CARD('b)$
**instantiation** *vec* :: (*zero*, *finite*) *zero*
**begin**

**instantiation** *vec* :: (*plus*, *finite*) *plus*
**begin**

**instantiation** *vec* :: (*minus*, *finite*) *minus*
**begin**

**instantiation** *vec* :: (*uminus*, *finite*) *uminus*
**begin**
**instantiation** *vec* :: (*times*, *finite*) *times*
**begin**

**instantiation** *vec* :: (*one*, *finite*) *one*
**begin**

**instantiation** *vec* :: (*ord*, *finite*) *ord*
**begin**

### 1.8.2 Real vector space

**definition** $scaleR \equiv (\lambda\ r\ x.\ (\chi\ i.\ scaleR\ r\ (x\$i)))$

### 1.8.3 Topological space

**definition** [*code del*]:
$open\ (S :: ('a\ \widehat{}\ 'b)\ set) \longleftrightarrow$
$(\forall\, x{\in}S.\ \exists\, A.\ (\forall\, i.\ open\ (A\ i) \wedge x\$i \in A\ i) \wedge$
$(\forall\, y.\ (\forall\, i.\ y\$i \in A\ i) \longrightarrow y \in S))$

### 1.8.4 Metric space

**definition**
$dist\ x\ y = L2\_set\ (\lambda i.\ dist\ (x\$i)\ (y\$i))\ UNIV$

**definition** [*code del*]:
$(uniformity :: (('a\widehat{}'b::\_) \times ('a\widehat{}'b::\_))\ filter) =$
$(INF\ e{\in}\{0\ <..\}.\ principal\ \{(x,\ y).\ dist\ x\ y < e\})$

**proposition** *dist_vec_nth_le*: $dist\ (x\ \$\ i)\ (y\ \$\ i) \leq dist\ x\ y$

### 1.8.5 Normed vector space

**definition** $norm\ x = L2\_set\ (\lambda i.\ norm\ (x\$i))\ UNIV$

**definition** $sgn\ (x::'a\widehat{}'b) = scaleR\ (inverse\ (norm\ x))\ x$

### 1.8.6 Inner product space

**definition** $inner\ x\ y = sum\ (\lambda i.\ inner\ (x\$i)\ (y\$i))\ UNIV$

### 1.8.7 Euclidean space

**definition** $axis\ k\ x = (\chi\ i.\ if\ i = k\ then\ x\ else\ 0)$

**definition** $Basis = (\bigcup i.\ \bigcup u{\in}Basis.\ \{axis\ i\ u\})$

**proposition** *DIM_cart* [*simp*]: $DIM('a\widehat{}'b) = CARD('b) * DIM('a)$

### 1.8.8 Matrix operations

**definition** *map_matrix*::('a ⇒ 'b) ⇒ (('a, 'i::finite)vec, 'j::finite) vec ⇒ (('b, 'i)vec, 'j) vec **where**
*map_matrix f x* = (χ i j. f (x $ i $ j))

**definition** *matrix_matrix_mult* :: ('a::semiring_1) ^'n^'m ⇒ 'a ^'p^'n ⇒ 'a ^ 'p ^'m
(**infixl** ** 70)
**where** m ** m' == (χ i j. sum (λk. ((m$i)$k) * ((m'$k)$j)) (UNIV :: 'n set)) ::'a ^ 'p ^'m

**definition** *matrix_vector_mult* :: ('a::semiring_1) ^'n^'m ⇒ 'a ^'n ⇒ 'a ^ 'm
(**infixl** *v 70)
**where** m *v x ≡ (χ i. sum (λj. ((m$i)$j) * (x$j)) (UNIV ::'n set)) :: 'a^'m

**definition** *vector_matrix_mult* :: 'a ^ 'm ⇒ ('a::semiring_1) ^'n^'m ⇒ 'a ^'n
(**infixl** v* 70)
**where** v v* m == (χ j. sum (λi. ((m$i)$j) * (v$i)) (UNIV :: 'm set)) :: 'a^'n

**proposition** *matrix_mul_assoc*: A ** (B ** C) = (A ** B) ** C

**proposition** *matrix_vector_mul_assoc*: A *v (B *v x) = (A ** B) *v x

**proposition** *scalar_matrix_assoc*:
**fixes** A :: ('a::real_algebra_1)^'m^'n
**shows** $k *_R (A ** B) = (k *_R A) ** B$

**proposition** *matrix_scalar_ac*:
**fixes** A :: ('a::real_algebra_1)^'m^'n
**shows** $A ** (k *_R B) = k *_R A ** B$
**definition** *matrix* :: ('a::{plus,times, one, zero}^'m ⇒ 'a ^ 'n) ⇒ 'a^'m^'n
**where** *matrix f* = (χ i j. (f(axis j 1))$i)

### 1.8.9 Inverse matrices (not necessarily square)

**definition**
*invertible*(A::'a::semiring_1^'n^'m) ⟷ (∃ A'::'a^'m^'n. A ** A' = mat 1 ∧ A' ** A = mat 1)

**definition**
*matrix_inv*(A:: 'a::semiring_1^'n^'m) =
(SOME A'::'a^'m^'n. A ** A' = mat 1 ∧ A' ** A = mat 1)

**proposition** *scalar_invertible_iff*:
**fixes** A :: ('a::real_algebra_1)^'m^'n
**assumes** k ≠ 0 **and** *invertible* A
**shows** *invertible* $(k *_R A)$ ⟷ k ≠ 0 ∧ *invertible* A

**proposition** *vector_scaleR_matrix_ac*:
  **fixes** *k* :: *real* **and** *x* :: *real^'n* **and** *A* :: *real^'m^'n*
  **shows** *x v** (*k* $*_R$ *A*) = *k* $*_R$ (*x v** *A*)

**end**

## 1.9 Linear Algebra on Finite Cartesian Products

**theory** *Cartesian_Space*
  **imports**
    *HOL−Combinatorics.Transposition*
    *Finite_Cartesian_Product*
    *Linear_Algebra*
**begin**

### 1.9.1 Some interesting theorems and interpretations

### 1.9.2 Rank of a matrix

**definition** *rank* :: *'a::field^'n^'m=>nat*
  **where** *row_rank_def_gen*: *rank A* ≡ *vec.dim*(*rows A*)

### 1.9.3 Orthogonality of a matrix

**definition** *orthogonal_matrix* (*Q::'a::semiring_1^'n^'n*) ⟷
  *transpose Q* ** *Q* = *mat 1* ∧ *Q* ** *transpose Q* = *mat 1*

**proposition** *orthogonal_matrix_mul*:
  **fixes** *A* :: *real ^'n^'n*
  **assumes** *orthogonal_matrix A orthogonal_matrix B*
  **shows** *orthogonal_matrix*(*A* ** *B*)

**proposition** *orthogonal_transformation_matrix*:
  **fixes** *f*:: *real^'n* ⇒ *real^'n*
  **shows** *orthogonal_transformation f* ⟷ *linear f* ∧ *orthogonal_matrix*(*matrix f*)
  (**is** *?lhs* ⟷ *?rhs*)

### 1.9.4 Finding an Orthogonal Matrix

**proposition** *orthogonal_matrix_exists_basis*:

**fixes** *a* :: *real*^'n
**assumes** *norm a* = *1*
**obtains** *A* **where** *orthogonal_matrix A A* ∗v (*axis k 1*) = *a*

**proposition** *orthogonal_transformation_exists*:
**fixes** *a b* :: *real*^'n
**assumes** *norm a* = *norm b*
**obtains** *f* **where** *orthogonal_transformation f f a* = *b*

### 1.9.5 Scaling and isometry

**proposition** *scaling_linear*:
**fixes** *f* :: 'a::real_inner ⇒ 'a::real_inner
**assumes** *f0*: *f 0* = *0*
**and** *fd*: ∀ *x y*. *dist* (*f x*) (*f y*) = *c* ∗ *dist x y*
**shows** *linear f*
**proposition** *orthogonal_transformation_isometry*:
*orthogonal_transformation f* ⟷ *f*(*0*::'a::real_inner) = (*0*::'a) ∧ (∀ *x y*. *dist*(*f x*) (*f y*) = *dist x y*)

### 1.9.6 Induction on matrix row operations

**end**

## 1.10 Traces and Determinants of Square Matrices

**theory** *Determinants*
**imports**
*HOL−Combinatorics.Permutations*
*Cartesian_Space*
**begin**

### 1.10.1 Trace

**definition** *trace* :: 'a::semiring_1^'n^'n ⇒ 'a
**where** *trace A* = *sum* (λ*i*. ((*A*$*i*)$*i*)) (*UNIV*::'n set)

#### Definition of determinant

**definition** *det*:: 'a::comm_ring_1^'n^'n ⇒ 'a **where**
*det A* =
*sum* (λ*p*. *of_int* (*sign p*) ∗ *prod* (λ*i*. *A*$*i*$*p i*) (*UNIV* :: 'n set))
{*p*. *p permutes* (*UNIV* :: 'n set)}
**proposition** *det_diagonal*:
**fixes** *A* :: 'a::comm_ring_1^'n^'n

**assumes** *ld*: $\bigwedge i\ j.\ i \neq j \Longrightarrow A\$i\$j = 0$
**shows** *det A* = *prod* ($\lambda i.\ A\$i\$i$) (*UNIV*::'*n set*)

**proposition** *det_matrix_scaleR* [*simp*]: $det\ (matrix\ (((*_R)\ r)) :: real^\frown{}'n^\frown{}'n) = r$
$\frown CARD('n::finite)$

**proposition** *det_mul*:
**fixes** $A\ B :: 'a::comm\_ring\_1^\frown{}'n^\frown{}'n$
**shows** $det\ (A ** B) = det\ A * det\ B$

### 1.10.2 Relation to invertibility

**proposition** *invertible_det_nz*:
**fixes** $A::'a::\{field\}^\frown{}'n^\frown{}'n$
**shows** $invertible\ A \longleftrightarrow det\ A \neq 0$

#### Invertibility of matrices and corresponding linear functions

### 1.10.3 Cramer's rule

**proposition** *cramer_lemma*:
**fixes** $A :: 'a::\{field\}^\frown{}'n^\frown{}'n$
**shows** $det((\chi\ i\ j.\ if\ j = k\ then\ (A *v\ x)\$i\ else\ A\$i\$j):: 'a::\{field\}^\frown{}'n^\frown{}'n) = x\$k$
$* det\ A$

**proposition** *cramer*:
**fixes** $A ::'a::\{field\}^\frown{}'n^\frown{}'n$
**assumes** *d0*: $det\ A \neq 0$
**shows** $A *v\ x = b \longleftrightarrow x = (\chi\ k.\ det(\chi\ i\ j.\ if\ j{=}k\ then\ b\$i\ else\ A\$i\$j)\ /\ det\ A)$

**proposition** *det_orthogonal_matrix*:
**fixes** $Q:: 'a::linordered\_idom^\frown{}'n^\frown{}'n$
**assumes** *oQ*: *orthogonal_matrix Q*
**shows** $det\ Q = 1 \lor det\ Q = -\ 1$

**proposition** *orthogonal_transformation_det* [*simp*]:
**fixes** $f :: real^\frown{}'n \Rightarrow real^\frown{}'n$
**shows** $orthogonal\_transformation\ f \Longrightarrow |det\ (matrix\ f)| = 1$

### 1.10.4 Rotation, reflection, rotoinversion

**definition** $rotation\_matrix\ Q \longleftrightarrow orthogonal\_matrix\ Q \land det\ Q = 1$
**definition** $rotoinversion\_matrix\ Q \longleftrightarrow orthogonal\_matrix\ Q \land det\ Q = -\ 1$

**end**

## 1.11 Operators involving abstract topology

**theory** *Abstract_Topology*
  **imports**
    *Complex_Main*
    *HOL−Library.Set_Idioms*
    *HOL−Library.FuncSet*
**begin**

### 1.11.1 General notion of a topology as a value

**definition** *istopology* :: ('*a set* ⇒ *bool*) ⇒ *bool* **where**
  *istopology* $L \equiv (\forall S\ T.\ L\ S \longrightarrow L\ T \longrightarrow L\ (S \cap T)) \wedge (\forall \mathcal{K}.\ (\forall K{\in}\mathcal{K}.\ L\ K) \longrightarrow L\ (\bigcup \mathcal{K}))$

**typedef** '*a topology* = {*L*::('*a set*) ⇒ *bool*. *istopology L*}
  **morphisms** *openin topology*
**proposition** *openin_clauses*:
  **fixes** *U* :: '*a topology*
  **shows**
    *openin U* {}
    $\bigwedge S\ T.$ *openin U S* ⟹ *openin U T* ⟹ *openin U* ($S \cap T$)
    $\bigwedge K.$ ($\forall S \in K.$ *openin U S*) ⟹ *openin U* ($\bigcup K$)
**definition** *closedin* :: '*a topology* ⇒ '*a set* ⇒ *bool* **where**
*closedin U S* ⟷ *S* ⊆ *topspace U* ∧ *openin U* (*topspace U* − *S*)

### 1.11.2 The discrete topology

### 1.11.3 Subspace topology

**definition** *subtopology* :: '*a topology* ⇒ '*a set* ⇒ '*a topology*
  **where** *subtopology U V* = *topology* ($\lambda T.\ \exists S.\ T = S \cap V \wedge$ *openin U S*)

### 1.11.4 The canonical topology from the underlying type class

**abbreviation** *euclidean* :: '*a*::*topological_space topology*
  **where** *euclidean* ≡ *topology open*

### 1.11.5 Basic "localization" results are handy for connectedness.

### 1.11.6 Derived set (set of limit points)

### 1.11.7 Closure with respect to a topological space

### 1.11.8 Frontier with respect to topological space

### 1.11.9 Locally finite collections

### 1.11.10 Continuous maps

**lemma** *continuous_map_alt*:
*continuous_map T1 T2 f*
$= ((\forall\ U.\ openin\ T2\ U \longrightarrow openin\ T1\ (f -` U \cap topspace\ T1)) \wedge f \in topspace\ T1 \rightarrow topspace\ T2)$

### 1.11.11 Open and closed maps (not a priori assumed continuous)

### 1.11.12 Quotient maps

### 1.11.13 Separated Sets

### 1.11.14 Homeomorphisms

### 1.11.15 Relation of homeomorphism between topological spaces

### 1.11.16 Connected topological spaces

### 1.11.17 Compact sets

**proposition** *compact_space_fip*:

$compact\_space\ X \longleftrightarrow$
$(\forall \mathcal{U}.\ (\forall C{\in}\mathcal{U}.\ closedin\ X\ C) \wedge (\forall \mathcal{F}.\ finite\ \mathcal{F} \wedge \mathcal{F} \subseteq \mathcal{U} \longrightarrow \bigcap \mathcal{F} \neq \{\}) \longrightarrow \bigcap \mathcal{U} \neq \{\})$
(**is** _ = *?rhs*)

**corollary** *compactin_fip*:
$compactin\ X\ S \longleftrightarrow$
$S \subseteq topspace\ X \wedge$
$(\forall \mathcal{U}.\ (\forall C{\in}\mathcal{U}.\ closedin\ X\ C) \wedge (\forall \mathcal{F}.\ finite\ \mathcal{F} \wedge \mathcal{F} \subseteq \mathcal{U} \longrightarrow S \cap \bigcap \mathcal{F} \neq \{\}) \longrightarrow S \cap \bigcap \mathcal{U} \neq \{\})$

**corollary** *compact_space_imp_nest*:
**fixes** $C :: nat \Rightarrow 'a\ set$
**assumes** *compact_space X* **and** *clo*: $\bigwedge n.\ closedin\ X\ (C\ n)$
**and** *ne*: $\bigwedge n.\ C\ n \neq \{\}$ **and** *dec*: *decseq C*
**shows** $(\bigcap n.\ C\ n) \neq \{\}$

### 1.11.18 Embedding maps

### 1.11.19 Retraction and section maps

### 1.11.20 Continuity

### 1.11.21 The topology generated by some (open) subsets

### 1.11.22 Topology bases and sub-bases

### 1.11.23 Continuity via bases/subbases, hence upper and lower semicontinuity

### 1.11.24 Pullback topology

**definition** $pullback\_topology::('a\ set) \Rightarrow ('a \Rightarrow 'b) \Rightarrow ('b\ topology) \Rightarrow ('a\ topology)$
**where** $pullback\_topology\ A\ f\ T = topology\ (\lambda S.\ \exists U.\ openin\ T\ U \wedge S = f -`U \cap A)$

**proposition** *continuous_map_pullback* [*intro*]:
**assumes** *continuous_map T1 T2 g*
**shows** *continuous_map (pullback_topology A f T1) T2 (g o f)*

**proposition** *continuous_map_pullback'* [*intro*]:
**assumes** $continuous\_map\ T1\ T2\ (f\ o\ g)\ topspace\ T1 \subseteq g -`A$
**shows** *continuous_map T1 (pullback_topology A f T2) g*

#### 1.11.25 Proper maps (not a priori assumed continuous)

#### 1.11.26 Perfect maps (proper, continuous and surjective)

**end**

## 1.12 $F$-Sigma and $G$-Delta sets in a Topological Space

**theory** *FSigma*
**imports** *Abstract_Topology*
**begin**

**end**

## 1.13 Disjoint sum of arbitarily many spaces

**theory** *Sum_Topology*
**imports** *Abstract_Topology*
**begin**

**end**

# Chapter 2

# Topology

**theory** *Elementary_Topology*
**imports**
*HOL−Library.Set_Idioms*
*HOL−Library.Disjoint_Sets*
*Product_Vector*
**begin**

## 2.1 Elementary Topology

### 2.1.1 Topological Basis

**definition** *topological_basis B* $\longleftrightarrow$
$(\forall b{\in}B.\ open\ b) \wedge (\forall x.\ open\ x \longrightarrow (\exists B'.\ B' \subseteq B \wedge \bigcup B' = x))$

### 2.1.2 Countable Basis

**locale** *countable_basis* = *topological_space p* **for** $p{::}'a\ set \Rightarrow bool$ +
**fixes** $B :: 'a\ set\ set$
**assumes** *is_basis*: *topological_basis B*
**and** *countable_basis*: *countable B*
**begin**

**class** *second_countable_topology* = *topological_space* +
**assumes** *ex_countable_subbasis*:
$\exists B{::}'a\ set\ set.\ countable\ B \wedge open = generate\_topology\ B$
**begin**

**proposition** *Lindelof*:
**fixes** $\mathcal{F} :: 'a{::}second\_countable\_topology\ set\ set$
**assumes** $\mathcal{F}$: $\bigwedge S.\ S \in \mathcal{F} \Longrightarrow open\ S$
**obtains** $\mathcal{F}'$ **where** $\mathcal{F}' \subseteq \mathcal{F}$ *countable* $\mathcal{F}'$ $\bigcup\mathcal{F}' = \bigcup\mathcal{F}$

### 2.1.3 Polish spaces

**class** *polish_space* = *complete_space* + *second_countable_topology*

### 2.1.4 Limit Points

**definition** (**in** *topological_space*) *islimpt*:: $'a \Rightarrow 'a\ set \Rightarrow bool$ (**infixr** *islimpt 60*)
**where** $x\ islimpt\ S \longleftrightarrow (\forall T.\ x{\in}T \longrightarrow open\ T \longrightarrow (\exists y{\in}S.\ y{\in}T \wedge y{\neq}x))$

### 2.1.5 Interior of a Set

**definition** *interior* :: $('a::topological\_space)\ set \Rightarrow 'a\ set$ **where**
$interior\ S = \bigcup\{T.\ open\ T \wedge T \subseteq S\}$

### 2.1.6 Closure of a Set

**definition** *closure* :: $('a::topological\_space)\ set \Rightarrow 'a\ set$ **where**
$closure\ S = S \cup \{x\ .\ x\ islimpt\ S\}$

### 2.1.7 Frontier (also known as boundary)

**definition** *frontier* :: $('a::topological\_space)\ set \Rightarrow 'a\ set$ **where**
$frontier\ S = closure\ S - interior\ S$

### 2.1.8 Limits

### 2.1.9 Compactness

**proposition** *Heine_Borel_imp_Bolzano_Weierstrass*:
**assumes** *compact S*
**and** *infinite T*
**and** $T \subseteq S$
**shows** $\exists x \in S.\ x\ islimpt\ T$

**definition** *countably_compact* :: $('a::topological\_space)\ set \Rightarrow bool$ **where**
$countably\_compact\ U \longleftrightarrow$
$(\forall A.\ countable\ A \longrightarrow (\forall a{\in}A.\ open\ a) \longrightarrow U \subseteq \bigcup A$
$\longrightarrow (\exists T{\subseteq}A.\ finite\ T \wedge U \subseteq \bigcup T))$

**proposition** *countably_compact_imp_compact_second_countable*:
$countably\_compact\ U \Longrightarrow compact\ (U :: 'a :: second\_countable\_topology\ set)$
**definition** *seq_compact* :: $'a::topological\_space\ set \Rightarrow bool$ **where**

*seq_compact S* $\longleftrightarrow$
  $(\forall f.\ (\forall n.\ f\ n \in S) \longrightarrow (\exists l{\in}S.\ \exists r{::}nat{\Rightarrow}nat.\ strict\_mono\ r \wedge (f \circ r) \longrightarrow l))$

**proposition** *Bolzano_Weierstrass_imp_seq_compact*:
  **fixes** $S :: 'a{::}\{t1\_space,\ first\_countable\_topology\}\ set$
  **shows** $(\bigwedge T.\ [\![infinite\ T;\ T \subseteq S]\!] \Longrightarrow \exists x \in S.\ x\ islimpt\ T) \Longrightarrow seq\_compact\ S$

### 2.1.10 Continuity

### 2.1.11 Homeomorphisms

**definition** *homeomorphism S T f g* $\longleftrightarrow$
  $(\forall x{\in}S.\ (g(f\ x) = x)) \wedge (f\ `\ S = T) \wedge continuous\_on\ S\ f \wedge$
  $(\forall y{\in}T.\ (f(g\ y) = y)) \wedge (g\ `\ T = S) \wedge continuous\_on\ T\ g$

**definition** *homeomorphic* $::\ 'a{::}topological\_space\ set \Rightarrow\ 'b{::}topological\_space\ set \Rightarrow bool$
  (**infixr** *homeomorphic 60*)
  **where** $s\ homeomorphic\ t \equiv (\exists f\ g.\ homeomorphism\ s\ t\ f\ g)$

**end**
**theory** *Abstract_Limits*
  **imports**
    *Abstract_Topology*
**begin**

### 2.1.12 nhdsin and atin

### 2.1.13 Limits in a topological space

### 2.1.14 Pointwise continuity in topological spaces

### 2.1.15 Combining theorems for continuous functions into the reals

**end**

## 2.2 Non-Denumerability of the Continuum

**theory** *Continuum_Not_Denumerable*
**imports**
  *Complex_Main*
  *HOL−Library.Countable_Set*

**begin**

**theorem** *real_non_denum*: $\nexists f$ :: *nat* ⇒ *real*. *surj f*

**corollary** *complex_non_denum*: $\nexists f$ :: *nat* ⇒ *complex*. *surj f*

**end**

## 2.3 Abstract Topology 2

**theory** *Abstract_Topology_2*
  **imports**
    *Elementary_Topology Abstract_Topology Continuum_Not_Denumerable*
    *HOL−Library.Indicator_Function*
    *HOL−Library.Equipollence*
**begin**

### 2.3.1 Closure

**corollary** *infinite_openin*:
  **fixes** *S* :: *'a* :: *t1_space set*
  **shows** ⟦*openin* (*top_of_set U*) *S*; *x* ∈ *S*; *x islimpt U*⟧ ⟹ *infinite S*

### 2.3.2 Frontier

### 2.3.3 Compactness

### 2.3.4 Continuity

### 2.3.5 Retractions

**definition** *retraction* :: (*'a*::*topological_space*) *set* ⇒ *'a set* ⇒ (*'a* ⇒ *'a*) ⇒ *bool*
**where** *retraction S T r* ⟷
  *T* ⊆ *S* ∧ *continuous_on S r* ∧ *r* ∈ *S* → *T* ∧ (∀ *x*∈*T*. *r x* = *x*)

**definition** *retract_of* (**infixl** *retract'_of 50*) **where**
*T retract_of S* ⟷ (∃ *r*. *retraction S T r*)

### 2.3.6 Retractions on a topological space

### 2.3.7 Paths and path-connectedness

### 2.3.8 Connected components

### 2.3.9 Combining theorems for continuous functions into the reals

### 2.3.10 A few cardinality results

**end**

## 2.4 Connected Components

**theory** *Connected*
**imports**
*Abstract_Topology_2*
**begin**

### 2.4.1 Connected components, considered as a connectedness relation or a set

**definition** *connected_component* $S\ x\ y \equiv \exists\, T.\ connected\ T \wedge T \subseteq S \wedge x \in T \wedge y \in T$

### 2.4.2 The set of connected components of a set

**definition** *components*:: $'a$::*topological_space set* $\Rightarrow$ $'a$ *set set*
**where** *components* $S \equiv$ *connected_component_set* $S$ ‘ $S$

### 2.4.3 Lemmas about components

**proposition** *component_diff_connected*:
**fixes** $S$ :: $'a$::*metric_space set*
**assumes** *connected* $S$ *connected* $U$ $S \subseteq U$ **and** $C$: $C \in$ *components* $(U - S)$
**shows** *connected*$(U - C)$

**end**

**theory** *Function_Topology*
**imports**
*Elementary_Topology*

*Abstract_Limits*
*Connected*
**begin**

## 2.5 Function Topology

### 2.5.1 The product topology

**definition** *product_topology*::('i ⇒ ('a topology)) ⇒ ('i set) ⇒ (('i ⇒ 'a) topology)
**where** *product_topology T I =*
*topology_generated_by* {($\Pi_E$ i∈I. X i) |X. (∀ i. openin (T i) (X i)) ∧ finite {i. X i ≠ topspace (T i)}}
**proposition** *product_topology*:
*product_topology X I =*
*topology*
(*arbitrary union_of*
((*finite intersection_of*
(λF. ∃ i U. F = {f. f i ∈ U} ∧ i ∈ I ∧ openin (X i) U))
*relative_to* ($\Pi_E$ i∈I. topspace (X i))))
(**is** _ = topology (_ union_of ((_ intersection_of ?Ψ) relative_to ?TOP)))

**proposition** *product_topology_open_contains_basis*:
**assumes** *openin* (*product_topology T I*) *U x* ∈ *U*
**shows** ∃ X. x ∈ ($\Pi_E$ i∈I. X i) ∧ (∀ i. openin (T i) (X i)) ∧ finite {i. X i ≠ topspace (T i)} ∧ ($\Pi_E$ i∈I. X i) ⊆ U

**corollary** *openin_product_topology_alt*:
*openin* (*product_topology X I*) *S* ⟷
(∀ x ∈ S. ∃ U. finite {i ∈ I. U i ≠ topspace(X i)} ∧
(∀ i ∈ I. openin (X i) (U i)) ∧ x ∈ $Pi_E$ I U ∧ $Pi_E$ I U ⊆ S)

**corollary** *closedin_product_topology*:
*closedin* (*product_topology X I*) (*PiE I S*) ⟷ *PiE I S* = {} ∨ (∀ i ∈ I. closedin (X i) (S i))

**corollary** *closedin_product_topology_singleton*:
*f* ∈ *extensional I* ⟹ *closedin* (*product_topology X I*) {f} ⟷ (∀ i ∈ I. closedin (X i) {f i})

#### Powers of a single topological space as a topological space, using type classes

**instantiation** *fun* :: (*type*, *topological_space*) *topological_space*
**begin**

**definition** *open_fun_def*:
*open U = openin* (*product_topology* (λi. euclidean) *UNIV*) *U*

**proposition** *product_topology_basis'*:
 **fixes** $x::'i \Rightarrow 'a$ **and** $U::'i \Rightarrow ('b::topological\_space)\ set$
 **assumes** $finite\ I\ \bigwedge i.\ i \in I \Longrightarrow open\ (U\ i)$
 **shows** $open\ \{f.\ \forall i{\in}I.\ f\ (x\ i) \in U\ i\}$

**Topological countability for product spaces**

**proposition** *product_topology_countable_basis*:
 **shows** $\exists K::(('a::countable \Rightarrow 'b::second\_countable\_topology)\ set\ set).$
 $topological\_basis\ K \wedge countable\ K \wedge$
 $(\forall k{\in}K.\ \exists X.\ (k = Pi_E\ UNIV\ X) \wedge (\forall i.\ open\ (X\ i)) \wedge finite\ \{i.\ X\ i \neq UNIV\})$

### 2.5.2 The Alexander subbase theorem

**theorem** *Alexander_subbase*:
 **assumes** $X$: $topology\ (arbitrary\ union\_of\ (finite\ intersection\_of\ (\lambda x.\ x \in \mathcal{B})\ relative\_to\ \bigcup\mathcal{B})) = X$
 **and** $fin$: $\bigwedge C.\ [\![C \subseteq \mathcal{B};\ \bigcup C = topspace\ X]\!] \Longrightarrow \exists C'.\ finite\ C' \wedge C' \subseteq C \wedge \bigcup C' = topspace\ X$
 **shows** $compact\_space\ X$

**corollary** *Alexander_subbase_alt*:
 **assumes** $U \subseteq \bigcup\mathcal{B}$
 **and** $fin$: $\bigwedge C.\ [\![C \subseteq \mathcal{B};\ U \subseteq \bigcup C]\!] \Longrightarrow \exists C'.\ finite\ C' \wedge C' \subseteq C \wedge U \subseteq \bigcup C'$
 **and** $X$: $topology$
 $(arbitrary\ union\_of$
 $(finite\ intersection\_of\ (\lambda x.\ x \in \mathcal{B})\ relative\_to\ U)) = X$
 **shows** $compact\_space\ X$

**proposition** *continuous_map_componentwise*:
 $continuous\_map\ X\ (product\_topology\ Y\ I)\ f \longleftrightarrow$
 $f\ `\ (topspace\ X) \subseteq extensional\ I \wedge (\forall k \in I.\ continuous\_map\ X\ (Y\ k)\ (\lambda x.\ f\ x\ k))$
 (**is** $?lhs \longleftrightarrow \_ \wedge ?rhs$)

**proposition** *open_map_product_projection*:
 **assumes** $i \in I$
 **shows** $open\_map\ (product\_topology\ Y\ I)\ (Y\ i)\ (\lambda f.\ f\ i)$

### 2.5.3 Open Pi-sets in the product topology

**proposition** *openin_PiE_gen*:
 $openin\ (product\_topology\ X\ I)\ (PiE\ I\ S) \longleftrightarrow$

*PiE I S* = {} ∨
*finite* {*i* ∈ *I*. *S i* ≠ *topspace* (*X i*)} ∧ (∀ *i* ∈ *I*. *openin* (*X i*) (*S i*))
(**is** *?lhs* ⟷ _ ∨ *?rhs*)

**corollary** *openin_PiE*:
*finite I* ⟹ *openin* (*product_topology X I*) (*PiE I S*) ⟷ *PiE I S* = {} ∨ (∀ *i* ∈ *I*. *openin* (*X i*) (*S i*))

**proposition** *compact_space_product_topology*:
*compact_space*(*product_topology X I*) ⟷
(*product_topology X I*) = *trivial_topology* ∨ (∀ *i* ∈ *I*. *compact_space*(*X i*))
(**is** *?lhs* = *?rhs*)

**corollary** *compactin_PiE*:
*compactin* (*product_topology X I*) (*PiE I S*) ⟷
*PiE I S* = {} ∨ (∀ *i* ∈ *I*. *compactin* (*X i*) (*S i*))

### 2.5.4 Relationship with connected spaces, paths, etc.

**proposition** *connected_space_product_topology*:
*connected_space*(*product_topology X I*) ⟷
(∃ *i* ∈ *I*. *X i* = *trivial_topology*) ∨ (∀ *i* ∈ *I*. *connected_space*(*X i*))
(**is** *?lhs* ⟷ *?eq* ∨ *?rhs*)

### 2.5.5 Projections from a function topology to a component

### 2.5.6 Limits

**end**

## 2.6 The binary product topology

**theory** *Product_Topology*
**imports** *Function_Topology*
**begin**

## 2.7 Product Topology

### 2.7.1 Definition

### 2.7.2 Continuity

**proposition** *compact_space_prod_topology*:
*compact_space*(*prod_topology X Y*) $\longleftrightarrow$ (*prod_topology X Y*) = *trivial_topology* $\vee$ *compact_space X* $\wedge$ *compact_space Y*

### 2.7.3 Homeomorphic maps

**proposition** *connected_space_prod_topology*:
*connected_space*(*prod_topology X Y*) $\longleftrightarrow$
(*prod_topology X Y*) = *trivial_topology* $\vee$ *connected_space X* $\wedge$ *connected_space Y* (**is** *?lhs=?rhs*)

**end**

## 2.8 T1 and Hausdorff spaces

**theory** *T1_Spaces*
**imports** *Product_Topology*
**begin**

## 2.9 T1 spaces with equivalences to many naturally "nice" properties.

**proposition** *t1_space_product_topology*:
*t1_space* (*product_topology X I*)
$\longleftrightarrow$ (*product_topology X I*) = *trivial_topology* $\vee$ ($\forall$ *i* $\in$ *I*. *t1_space* (*X i*))

### 2.9.1 Hausdorff Spaces

**end**

## 2.10 Lindelöf spaces

**theory** *Lindelof_Spaces*
**imports** *T1_Spaces*
**begin**

**end**

# Chapter 3

# Functional Analysis

**theory** *Metric_Arith*
  **imports** *HOL.Real_Vector_Spaces*
**begin**
**theorem** *metric_eq_thm* [*THEN HOL.eq_reflection*]:
  $x \in s \Longrightarrow y \in s \Longrightarrow x = y \longleftrightarrow (\forall\, a{\in}s.\ dist\ x\ a = dist\ y\ a)$

**end**

## 3.1 Elementary Metric Spaces

**theory** *Elementary_Metric_Spaces*
  **imports**
    *Abstract_Topology_2*
    *Metric_Arith*
**begin**

### 3.1.1 Open and closed balls

**definition** *ball* :: $'a{::}metric\_space \Rightarrow real \Rightarrow 'a\ set$
  **where** $ball\ x\ e = \{y.\ dist\ x\ y < e\}$

**definition** *cball* :: $'a{::}metric\_space \Rightarrow real \Rightarrow 'a\ set$
  **where** $cball\ x\ e = \{y.\ dist\ x\ y \le e\}$

**definition** *sphere* :: $'a{::}metric\_space \Rightarrow real \Rightarrow 'a\ set$
  **where** $sphere\ x\ e = \{y.\ dist\ x\ y = e\}$

### 3.1.2 Limit Points

### 3.1.3 Perfect Metric Spaces

### 3.1.4 Finite and discrete

### 3.1.5 Interior

### 3.1.6 Frontier

### 3.1.7 Limits

**proposition** *Lim*: $(f \longrightarrow l)\ net \longleftrightarrow trivial\_limit\ net \lor (\forall\, e{>}0.\ eventually\ (\lambda x.\ dist\ (f\,x)\ l < e)\ net)$
**proposition** *Lim_within_le*: $(f \longrightarrow l)(at\ a\ within\ S) \longleftrightarrow$
$(\forall\, e{>}0.\ \exists\, d{>}0.\ \forall\, x{\in}S.\ 0 < dist\ x\ a \land dist\ x\ a \leq d \longrightarrow dist\ (f\,x)\ l < e)$

**proposition** *Lim_within*: $(f \longrightarrow l)\ (at\ a\ within\ S) \longleftrightarrow$
$(\forall\, e > 0.\ \exists\, d{>}0.\ \forall\, x \in S.\ 0 < dist\ x\ a \land dist\ x\ a < d \longrightarrow dist\ (f\,x)\ l < e)$

**corollary** *Lim_withinI* [*intro?*]:
**assumes** $\bigwedge e.\ e > 0 \Longrightarrow \exists\, d{>}0.\ \forall\, x \in S.\ 0 < dist\ x\ a \land dist\ x\ a < d \longrightarrow dist\ (f\,x)\ l \leq e$
**shows** $(f \longrightarrow l)\ (at\ a\ within\ S)$

**proposition** *Lim_at*: $(f \longrightarrow l)\ (at\ a) \longleftrightarrow$
$(\forall\, e > 0.\ \exists\, d{>}0.\ \forall\, x.\ 0 < dist\ x\ a \land dist\ x\ a < d \longrightarrow dist\ (f\,x)\ l < e)$

### 3.1.8 Continuity

**proposition** *continuous_within_eps_delta*:
$continuous\ (at\ x\ within\ s)\ f \longleftrightarrow (\forall\, e{>}0.\ \exists\, d{>}0.\ \forall\, x'{\in}\ s.\ dist\ x'\ x < d\ {-}{-}{>} dist\ (f\,x')\ (f\,x) < e)$

**corollary** *continuous_at_eps_delta*:
$continuous\ (at\ x)\ f \longleftrightarrow (\forall\, e > 0.\ \exists\, d > 0.\ \forall\, x'.\ dist\ x'\ x < d \longrightarrow dist\ (f\,x')\ (f\,x) < e)$

### 3.1.9 Closure and Limit Characterization

### 3.1.10 Boundedness

**definition** (**in** *metric_space*) $bounded :: 'a\ set \Rightarrow bool$
**where** $bounded\ S \longleftrightarrow (\exists\, x\ e.\ \forall\, y{\in}S.\ dist\ x\ y \leq e)$

### 3.1.11 Compactness

**proposition** *seq_compact_imp_totally_bounded*:
**assumes** *seq_compact S*

**shows** $\forall e>0. \exists k. finite\ k \wedge k \subseteq S \wedge S \subseteq (\bigcup x\in k.\ ball\ x\ e)$
**proposition** *seq_compact_imp_Heine_Borel*:
**fixes** $S$ :: *'a :: metric_space set*
**assumes** *seq_compact S*
**shows** *compact S*

**proposition** *compact_eq_seq_compact_metric*:
*compact* ($S$ :: *'a::metric_space set*) $\longleftrightarrow$ *seq_compact S*

**proposition** *compact_def*: — this is the definition of compactness in HOL Light
*compact* ($S$ :: *'a::metric_space set*) $\longleftrightarrow$
$(\forall f. (\forall n. f\ n \in S) \longrightarrow (\exists l\in S. \exists r::nat\Rightarrow nat.\ strict\_mono\ r \wedge (f \circ r) \longlongrightarrow l))$
**proposition** *compact_eq_Bolzano_Weierstrass*:
**fixes** $S$ :: *'a::metric_space set*
**shows** *compact S* $\longleftrightarrow (\forall T.\ infinite\ T \wedge T \subseteq S \longrightarrow (\exists x \in S.\ x\ islimpt\ T))$

**proposition** *Bolzano_Weierstrass_imp_bounded*:
$(\bigwedge T. [\![infinite\ T;\ T \subseteq S]\!] \Longrightarrow (\exists x \in S.\ x\ islimpt\ T)) \Longrightarrow bounded\ S$

### 3.1.12 Banach fixed point theorem

**theorem** *banach_fix*:— TODO: rename to *Banach_fix*
**assumes** *s*: *complete s* $s \neq \{\}$
**and** *c*: $0 \leq c$ $c < 1$
**and** *f*: $f$ ‘ $s \subseteq s$
**and** *lipschitz*: $\forall x\in s. \forall y\in s.\ dist\ (f\ x)\ (f\ y) \leq c * dist\ x\ y$
**shows** $\exists! x\in s.\ f\ x = x$

### 3.1.13 Edelstein fixed point theorem

**theorem** *Edelstein_fix*:
**fixes** $S$ :: *'a::metric_space set*
**assumes** *S*: *compact S* $S \neq \{\}$
**and** *gs*: $(g$ ‘ $S) \subseteq S$
**and** *dist*: $\forall x\in S. \forall y\in S.\ x \neq y \longrightarrow dist\ (g\ x)\ (g\ y) < dist\ x\ y$
**shows** $\exists! x\in S.\ g\ x = x$

### 3.1.14 The diameter of a set

**definition** *diameter* :: *'a::metric_space set* $\Rightarrow$ *real* **where**
*diameter S* $=$ (*if* $S = \{\}$ *then* 0 *else SUP* $(x,y)\in S\times S.\ dist\ x\ y)$

**proposition** *Lebesgue_number_lemma*:
**assumes** *compact S* $\mathcal{C} \neq \{\}$ $S \subseteq \bigcup\mathcal{C}$ **and** *ope*: $\bigwedge B.\ B \in \mathcal{C} \Longrightarrow open\ B$
**obtains** $\delta$ **where** $0 < \delta$ $\bigwedge T. [\![T \subseteq S;\ diameter\ T < \delta]\!] \Longrightarrow \exists B \in \mathcal{C}.\ T \subseteq B$

### 3.1.15 Metric spaces with the Heine-Borel property

**class** *heine_borel* = *metric_space* +
  **assumes** *bounded_imp_convergent_subsequence*:
    $bounded\ (range\ f) \Longrightarrow \exists\, l\ r.\ strict\_mono\ (r{::}nat{\Rightarrow}nat) \wedge ((f \circ r) \longrightarrow l)$
*sequentially*

**proposition** *bounded_closed_imp_seq_compact*:
  **fixes** $S{::}'a{::}heine\_borel\ set$
  **assumes** *bounded S*
   **and** *closed S*
  **shows** *seq_compact S*

**instance** *real* :: *heine_borel*

**instance** *prod* :: (*heine_borel*, *heine_borel*) *heine_borel*

### 3.1.16 Completeness

**proposition** (**in** *metric_space*) *completeI*:
  **assumes** $\bigwedge f.\ \forall\, n.\ f\ n \in s \Longrightarrow Cauchy\ f \Longrightarrow \exists\, l{\in}s.\ f \longrightarrow l$
  **shows** *complete s*

**proposition** (**in** *metric_space*) *completeE*:
  **assumes** *complete s* **and** $\forall\, n.\ f\ n \in s$ **and** *Cauchy f*
  **obtains** *l* **where** $l \in s$ **and** $f \longrightarrow l$

**proposition** *compact_eq_totally_bounded*:
  $compact\ s \longleftrightarrow complete\ s \wedge (\forall\, e{>}0.\ \exists\, k.\ finite\ k \wedge s \subseteq (\bigcup x{\in}k.\ ball\ x\ e))$
   (**is** _ $\longleftrightarrow$ *?rhs*)

### 3.1.17 Cauchy continuity

### 3.1.18 Properties of Balls and Spheres

### 3.1.19 Distance from a Set

### 3.1.20 Infimum Distance

**definition** $infdist\ x\ A = (if\ A = \{\}\ then\ 0\ else\ INF\ a{\in}A.\ dist\ x\ a)$

### 3.1.21 Separation between Points and Sets

**proposition** *separate_point_closed*:
  **fixes** $S :: 'a{::}heine\_borel\ set$
  **assumes** *closed S* **and** $a \notin S$
  **shows** $\exists\, d{>}0.\ \forall\, x{\in}S.\ d \leq dist\ a\ x$

**proposition** *separate_compact_closed*:
  **fixes** $S$ $T$ :: *'a::heine_borel set*
  **assumes** *compact* $S$
    **and** $T$: *closed* $T$ $S \cap T = \{\}$
  **shows** $\exists d>0.\ \forall x\in S.\ \forall y\in T.\ d \leq dist\ x\ y$

**proposition** *separate_closed_compact*:
  **fixes** $S$ $T$ :: *'a::heine_borel set*
  **assumes** $S$: *closed* $S$
    **and** $T$: *compact* $T$
    **and** *dis*: $S \cap T = \{\}$
  **shows** $\exists d>0.\ \forall x\in S.\ \forall y\in T.\ d \leq dist\ x\ y$

**proposition** *compact_in_open_separated*:
  **fixes** *A::'a::heine_borel set*
  **assumes** $A$: $A \neq \{\}$ *compact* $A$
  **assumes** *open* $B$
  **assumes** $A \subseteq B$
  **obtains** $e$ **where** $e > 0$ $\{x.\ infdist\ x\ A \leq e\} \subseteq B$

### 3.1.22 Uniform Continuity

### 3.1.23 Continuity on a Compact Domain Implies Uniform Continuity

**corollary** *compact_uniformly_continuous*:
  **fixes** $f$ :: *'a :: metric_space ⇒ 'b :: metric_space*
  **assumes** $f$: *continuous_on* $S$ $f$ **and** $S$: *compact* $S$
  **shows** *uniformly_continuous_on* $S$ $f$

### 3.1.24 With Abstract Topology (TODO: move and remove dependency?)

### 3.1.25 Closed Nest

### 3.1.26 Consequences for Real Numbers

### 3.1.27 The infimum of the distance between two sets

**definition** *setdist* :: *'a::metric_space set ⇒ 'a set ⇒ real* **where**
  *setdist s t* ≡
    (*if* $s = \{\} \vee t = \{\}$ *then 0*
     *else Inf* $\{dist\ x\ y |\ x\ y.\ x \in s \wedge y \in t\}$)

**proposition** *setdist_attains_inf*:
  **assumes** *compact* $B$ $B \neq \{\}$

**obtains** *y* **where** $y \in B$ *setdist A B* = *infdist y A*

**end**

## 3.2 Elementary Normed Vector Spaces

**theory** *Elementary_Normed_Spaces*
**imports**
*HOL−Library.FuncSet*
*Elementary_Metric_Spaces Cartesian_Space*
*Connected*
**begin**

### 3.2.1 Orthogonal Transformation of Balls

### 3.2.2 Support

### 3.2.3 Intervals

### 3.2.4 Limit Points

### 3.2.5 Balls and Spheres in Normed Spaces

**corollary** *compact_sphere* [*simp*]:
**fixes** *a* :: *'a*::{*real_normed_vector*,*perfect_space*,*heine_borel*}
**shows** *compact* (*sphere a r*)

**corollary** *bounded_sphere* [*simp*]:
**fixes** *a* :: *'a*::{*real_normed_vector*,*perfect_space*,*heine_borel*}
**shows** *bounded* (*sphere a r*)

**corollary** *closed_sphere* [*simp*]:
**fixes** *a* :: *'a*::{*real_normed_vector*,*perfect_space*,*heine_borel*}
**shows** *closed* (*sphere a r*)

### 3.2.6 Filters

### 3.2.7 Trivial Limits

### 3.2.8 Limits

**proposition** *Lim_at_infinity*: ($f \longrightarrow l$) *at_infinity* $\longleftrightarrow$ ($\forall e>0.\ \exists b.\ \forall x.$ *norm* $x \geq b \longrightarrow$ *dist* $(f\ x)\ l < e$)

**corollary** *Lim_at_infinityI* [*intro?*]:

**assumes** $\bigwedge e.\ e > 0 \Longrightarrow \exists B.\ \forall x.\ norm\ x \geq B \longrightarrow dist\ (f\ x)\ l \leq e$
**shows** $(f \longlongrightarrow l)\ at\_infinity$

### 3.2.9 Boundedness

**corollary** *cobounded_imp_unbounded*:
**fixes** $S :: 'a::\{real\_normed\_vector, perfect\_space\}\ set$
**shows** $bounded\ (-\ S) \Longrightarrow \neg\ bounded\ S$

### 3.2.10 Normed spaces with the Heine-Borel property

### 3.2.11 Intersecting chains of compact sets and the Baire property

**proposition** *bounded_closed_chain*:
**fixes** $\mathcal{F} :: 'a::heine\_borel\ set\ set$
**assumes** $B \in \mathcal{F}\ bounded\ B$ **and** $\mathcal{F}$: $\bigwedge S.\ S \in \mathcal{F} \Longrightarrow closed\ S$ **and** $\{\} \notin \mathcal{F}$
**and** *chain*: $\bigwedge S\ T.\ S \in \mathcal{F} \wedge T \in \mathcal{F} \Longrightarrow S \subseteq T \vee T \subseteq S$
**shows** $\bigcap \mathcal{F} \neq \{\}$

**corollary** *compact_chain*:
**fixes** $\mathcal{F} :: 'a::heine\_borel\ set\ set$
**assumes** $\bigwedge S.\ S \in \mathcal{F} \Longrightarrow compact\ S\ \{\} \notin \mathcal{F}$
$\bigwedge S\ T.\ S \in \mathcal{F} \wedge T \in \mathcal{F} \Longrightarrow S \subseteq T \vee T \subseteq S$
**shows** $\bigcap\ \mathcal{F} \neq \{\}$
**theorem** *Baire*:
**fixes** $S::'a::\{real\_normed\_vector,heine\_borel\}\ set$
**assumes** $closed\ S\ countable\ \mathcal{G}$
**and** *ope*: $\bigwedge T.\ T \in \mathcal{G} \Longrightarrow openin\ (top\_of\_set\ S)\ T \wedge S \subseteq closure\ T$
**shows** $S \subseteq closure(\bigcap \mathcal{G})$

### 3.2.12 Continuity

**proposition** *homeomorphic_ball_UNIV*:
**fixes** $a\ ::'a::real\_normed\_vector$
**assumes** $0 < r$ **shows** $ball\ a\ r\ homeomorphic\ (UNIV::\ 'a\ set)$

### 3.2.13 Connected Normed Spaces

**end**

## 3.3 Linear Decision Procedure for Normed Spaces

**theory** *Norm_Arith*
**imports** *HOL−Library.Sum_of_Squares*
**begin**

**method_setup** *norm* = ‹
  *Scan.succeed* (*SIMPLE_METHOD′ o NormArith.norm_arith_tac*)
› *prove simple linear statements about vector norms*

**proposition** *dist_triangle_add*:
  **fixes** $x\ y\ x'\ y'$ :: ′*a*::*real_normed_vector*
  **shows** *dist* $(x + y)\ (x' + y') \leq$ *dist* $x\ x'$ + *dist* $y\ y'$

**end**

# Chapter 4

# Vector Analysis

**theory** *Topology_Euclidean_Space*
**imports**
*Elementary_Normed_Spaces*
*Linear_Algebra*
*Norm_Arith*
**begin**

## 4.1 Elementary Topology in Euclidean Space

### 4.1.1 Boxes

**abbreviation** *One* :: $'a$::*euclidean_space* **where**
$One \equiv \sum Basis$

**definition** (**in** *euclidean_space*) *eucl_less* (**infix** <*e 50*) **where**
$eucl\_less\ a\ b \longleftrightarrow (\forall i \in Basis.\ a \cdot i < b \cdot i)$

**definition** *box_eucl_less*: $box\ a\ b = \{x.\ a <e\ x \wedge x <e\ b\}$
**definition** $cbox\ a\ b = \{x.\ \forall i \in Basis.\ a \cdot i \le x \cdot i \wedge x \cdot i \le b \cdot i\}$

**corollary** *open_countable_Union_open_box*:
**fixes** $S$ :: $'a$ :: *euclidean_space set*
**assumes** *open S*
**obtains** $\mathcal{D}$ **where** $countable\ \mathcal{D}\ \mathcal{D} \subseteq Pow\ S\ \bigwedge X.\ X \in \mathcal{D} \Longrightarrow \exists\ a\ b.\ X = box\ a\ b$
$\bigcup \mathcal{D} = S$

**corollary** *open_countable_Union_open_cbox*:
**fixes** $S$ :: $'a$ :: *euclidean_space set*
**assumes** *open S*
**obtains** $\mathcal{D}$ **where** $countable\ \mathcal{D}\ \mathcal{D} \subseteq Pow\ S\ \bigwedge X.\ X \in \mathcal{D} \Longrightarrow \exists\ a\ b.\ X = cbox\ a$
$b\ \bigcup \mathcal{D} = S$

### 4.1.2 General Intervals

**definition** *is_interval* $(s::('a::euclidean\_space)\ set) \longleftrightarrow$
$(\forall\, a{\in}s.\ \forall\, b{\in}s.\ \forall\, x.\ (\forall\, i{\in}Basis.\ ((a{\bullet}i \le x{\bullet}i \wedge x{\bullet}i \le b{\bullet}i) \vee (b{\bullet}i \le x{\bullet}i \wedge x{\bullet}i \le a{\bullet}i)))$
$\longrightarrow x \in s)$

### 4.1.3 Limit Component Bounds

### 4.1.4 Class Instances

**instance** *euclidean_space* ⊆ *heine_borel*

**instance** *euclidean_space* ⊆ *banach*

### 4.1.5 Compact Boxes

**proposition** *is_interval_compact*:
$is\_interval\ S \wedge compact\ S \longleftrightarrow (\exists\, a\ b.\ S = cbox\ a\ b)$ (**is** *?lhs = ?rhs*)

**proposition** *tendsto_componentwise_iff*:
**fixes** $f :: \_ \Rightarrow 'b::euclidean\_space$
**shows** $(f \longrightarrow l)\ F \longleftrightarrow (\forall\, i \in Basis.\ ((\lambda x.\ (f\ x \cdot i)) \longrightarrow (l \cdot i))\ F)$
(**is** *?lhs = ?rhs*)

**corollary** *continuous_componentwise*:
$continuous\ F\ f \longleftrightarrow (\forall\, i \in Basis.\ continuous\ F\ (\lambda x.\ (f\ x \cdot i)))$

**corollary** *continuous_on_componentwise*:
**fixes** $S :: 'a :: t2\_space\ set$
**shows** $continuous\_on\ S\ f \longleftrightarrow (\forall\, i \in Basis.\ continuous\_on\ S\ (\lambda x.\ (f\ x \cdot i)))$

### 4.1.6 Separability

**proposition** *separable*:
**fixes** $S :: 'a::\{metric\_space, second\_countable\_topology\}\ set$
**obtains** $T$ **where** $countable\ T\ T \subseteq S\ S \subseteq closure\ T$
**proposition** *open_surjective_linear_image*:
**fixes** $f :: 'a::real\_normed\_vector \Rightarrow 'b::euclidean\_space$
**assumes** $open\ A\ linear\ f\ surj\ f$
**shows** $open(f\ `\ A)$

**corollary** *open_bijective_linear_image_eq*:
  **fixes** $f$ :: $'a$::*euclidean_space* ⇒ $'b$::*euclidean_space*
  **assumes** *linear f bij f*
    **shows** $open(f$ ‘ $A) \longleftrightarrow open\ A$

**corollary** *interior_bijective_linear_image*:
  **fixes** $f$ :: $'a$::*euclidean_space* ⇒ $'b$::*euclidean_space*
  **assumes** *linear f bij f*
  **shows** $interior\ (f$ ‘ $S) = f$ ‘ $interior\ S$
**proposition** *injective_imp_isometric*:
  **fixes** $f$ :: $'a$::*euclidean_space* ⇒ $'b$::*euclidean_space*
  **assumes** $s$: *closed s subspace s*
    **and** $f$: *bounded_linear f* $\forall x{\in}s.\ f\ x = 0 \longrightarrow x = 0$
  **shows** $\exists e{>}0.\ \forall x{\in}s.\ norm\ (f\ x) \geq e * norm\ x$

**proposition** *closed_injective_image_subspace*:
  **fixes** $f$ :: $'a$::*euclidean_space* ⇒ $'b$::*euclidean_space*
  **assumes** *subspace s bounded_linear f* $\forall x{\in}s.\ f\ x = 0 \longrightarrow x = 0$ *closed s*
  **shows** $closed(f$ ‘ $s)$

### 4.1.7 Set Distance

**corollary** *setdist_gt_0_compact_closed*:
  **assumes** $S$: *compact S* **and** $T$: *closed T*
    **shows** $setdist\ S\ T > 0 \longleftrightarrow (S \neq \{\} \wedge T \neq \{\} \wedge S \cap T = \{\})$

**end**

## 4.2 Convex Sets and Functions on (Normed) Euclidean Spaces

**theory** *Convex_Euclidean_Space*
**imports**
  *Convex*
  *Topology_Euclidean_Space*
**begin**
**corollary** *empty_interior_lowdim*:
  **fixes** $S$ :: $'n$::*euclidean_space set*
  **shows** $dim\ S < DIM\ ('n) \Longrightarrow interior\ S = \{\}$

**corollary** *aff_dim_nonempty_interior*:
  **fixes** $S$ :: $'a$::*euclidean_space set*
  **shows** $interior\ S \neq \{\} \Longrightarrow aff\_dim\ S = DIM('a)$

### 4.2.1 Relative interior of a set

**definition** *rel_interior S* =
{*x*. ∃ *T*. *openin* (*top_of_set* (*affine hull S*)) *T* ∧ *x* ∈ *T* ∧ *T* ⊆ *S*}
**definition** *rel_open S* ⟷ *rel_interior S* = *S*

### 4.2.2 Closest point of a convex set is unique, with a continuous projection

**definition** *closest_point* :: ′*a*::{*real_inner*,*heine_borel*} *set* ⇒ ′*a* ⇒ ′*a*
**where** *closest_point S a* = (*SOME x*. *x* ∈ *S* ∧ (∀ *y*∈*S*. *dist a x* ≤ *dist a y*))

**proposition** *closest_point_in_rel_interior*:
**assumes** *closed S S* ≠ {} **and** *x*: *x* ∈ *affine hull S*
**shows** *closest_point S x* ∈ *rel_interior S* ⟷ *x* ∈ *rel_interior S*

**end**

## 4.3 Line Segment

**theory** *Line_Segment*
**imports**
*Convex*
*Topology_Euclidean_Space*
**begin**

**corollary** *component_complement_connected*:
**fixes** *S* :: ′*a*::*real_normed_vector set*
**assumes** *connected S C* ∈ *components* (−*S*)
**shows** *connected*(−*C*)

**proposition** *clopen*:
**fixes** *S* :: ′*a* :: *real_normed_vector set*
**shows** *closed S* ∧ *open S* ⟷ *S* = {} ∨ *S* = *UNIV*

**corollary** *compact_open*:
**fixes** *S* :: ′*a* :: *euclidean_space set*
**shows** *compact S* ∧ *open S* ⟷ *S* = {}

**corollary** *finite_imp_not_open*:
**fixes** *S* :: ′*a*::{*real_normed_vector*, *perfect_space*} *set*
**shows** ⟦*finite S*; *open S*⟧ ⟹ *S*={}

**corollary** *empty_interior_finite*:
**fixes** *S* :: ′*a*::{*real_normed_vector*, *perfect_space*} *set*
**shows** *finite S* ⟹ *interior S* = {}

### 4.3.1 Midpoint

**definition** *midpoint* :: $'a$::*real_vector* ⇒ $'a$ ⇒ $'a$
**where** *midpoint a b* = (*inverse* (*2*::*real*)) $*_R$ (*a* + *b*)

### 4.3.2 Open and closed segments

**definition** *closed_segment* :: $'a$::*real_vector* ⇒ $'a$ ⇒ $'a$ *set*
**where** *closed_segment a b* = {(*1* − *u*) $*_R$ *a* + *u* $*_R$ *b* | *u*::*real*. *0* ≤ *u* ∧ *u* ≤ *1*}

**definition** *open_segment* :: $'a$::*real_vector* ⇒ $'a$ ⇒ $'a$ *set* **where**
*open_segment a b* ≡ *closed_segment a b* − {*a*,*b*}

**proposition** *dist_decreases_open_segment*:
**fixes** *a* :: $'a$ :: *euclidean_space*
**assumes** *x* ∈ *open_segment a b*
**shows** *dist c x* < *dist c a* ∨ *dist c x* < *dist c b*

**corollary** *open_segment_furthest_le*:
**fixes** *a b x y* :: $'a$::*euclidean_space*
**assumes** *x* ∈ *open_segment a b*
**shows** *norm* (*y* − *x*) < *norm* (*y* − *a*) ∨ *norm* (*y* − *x*) < *norm* (*y* − *b*)

**corollary** *dist_decreases_closed_segment*:
**fixes** *a* :: $'a$ :: *euclidean_space*
**assumes** *x* ∈ *closed_segment a b*
**shows** *dist c x* ≤ *dist c a* ∨ *dist c x* ≤ *dist c b*

**corollary** *segment_furthest_le*:
**fixes** *a b x y* :: $'a$::*euclidean_space*
**assumes** *x* ∈ *closed_segment a b*
**shows** *norm* (*y* − *x*) ≤ *norm* (*y* − *a*) ∨ *norm* (*y* − *x*) ≤ *norm* (*y* − *b*)

### 4.3.3 Betweenness

**definition** *between* = (λ(*a*,*b*) *x*. *x* ∈ *closed_segment a b*)

**end**

# Chapter 5

# Unsorted

**theory** *Starlike*
  **imports**
    *Convex_Euclidean_Space*
    *Line_Segment*
**begin**

### 5.0.1 The relative frontier of a set

**definition** *rel_frontier S = closure S − rel_interior S*

**proposition** *ray_to_rel_frontier*:
  **fixes** *a* :: *'a::real_inner*
  **assumes** *bounded S*
    **and** *a*: *a ∈ rel_interior S*
    **and** *aff*: *(a + l) ∈ affine hull S*
    **and** *l ≠ 0*
  **obtains** *d* **where** *0 < d (a + d ∗R l) ∈ rel_frontier S*
    *⋀e. ⟦0 ≤ e; e < d⟧ ⟹ (a + e ∗R l) ∈ rel_interior S*

**corollary** *ray_to_frontier*:
  **fixes** *a* :: *'a::euclidean_space*
  **assumes** *bounded S*
    **and** *a*: *a ∈ interior S*
    **and** *l ≠ 0*
  **obtains** *d* **where** *0 < d (a + d ∗R l) ∈ frontier S*
    *⋀e. ⟦0 ≤ e; e < d⟧ ⟹ (a + e ∗R l) ∈ interior S*

**proposition** *rel_frontier_not_sing*:
  **fixes** *a* :: *'a::euclidean_space*
  **assumes** *bounded S*
    **shows** *rel_frontier S ≠ {a}*

### 5.0.2 Coplanarity, and collinearity in terms of affine hull

**definition** *coplanar* **where**
*coplanar* $S \equiv \exists\, u\ v\ w.\ S \subseteq$ *affine hull* $\{u,v,w\}$

### 5.0.3 Connectedness of the intersection of a chain

**proposition** *connected_chain*:
**fixes** $\mathcal{F}$ :: *'a :: euclidean_space set set*
**assumes** *cc*: $\bigwedge S.\ S \in \mathcal{F} \Longrightarrow$ *compact* $S \wedge$ *connected* $S$
**and** *linear*: $\bigwedge S\ T.\ S \in \mathcal{F} \wedge T \in \mathcal{F} \Longrightarrow S \subseteq T \vee T \subseteq S$
**shows** *connected*$(\bigcap \mathcal{F})$

### 5.0.4 Proper maps, including projections out of compact sets

**proposition** *proper_map*:
**fixes** *f* :: *'a::heine_borel* ⇒ *'b::heine_borel*
**assumes** *closedin* (*top_of_set S*) *K*
**and** *com*: $\bigwedge U.\ [\![U \subseteq T;$ *compact* $U]\!] \Longrightarrow$ *compact* $(S \cap f -` U)$
**and** $f$ ` $S \subseteq T$
**shows** *closedin* (*top_of_set T*) (*f* ` *K*)

**corollary** *affine_hull_convex_Int_open*:
**fixes** *S* :: *'a::real_normed_vector set*
**assumes** *convex S open T* $S \cap T \neq \{\}$
**shows** *affine hull* $(S \cap T) =$ *affine hull S*

**corollary** *affine_hull_affine_Int_nonempty_interior*:
**fixes** *S* :: *'a::real_normed_vector set*
**assumes** *affine S* $S \cap$ *interior* $T \neq \{\}$
**shows** *affine hull* $(S \cap T) =$ *affine hull S*

**corollary** *affine_hull_affine_Int_open*:
**fixes** *S* :: *'a::real_normed_vector set*
**assumes** *affine S open T* $S \cap T \neq \{\}$
**shows** *affine hull* $(S \cap T) =$ *affine hull S*

**corollary** *affine_hull_convex_Int_openin*:
**fixes** *S* :: *'a::real_normed_vector set*
**assumes** *convex S openin* (*top_of_set* (*affine hull S*)) *T* $S \cap T \neq \{\}$
**shows** *affine hull* $(S \cap T) =$ *affine hull S*

**corollary** *affine_hull_openin*:

**fixes** $S$ :: *'a::real_normed_vector set*
**assumes** *openin* (*top_of_set* (*affine hull* $T$)) $S$ $S \neq \{\}$
**shows** *affine hull* $S$ = *affine hull* $T$

**corollary** *affine_hull_open*:
**fixes** $S$ :: *'a::real_normed_vector set*
**assumes** *open* $S$ $S \neq \{\}$
**shows** *affine hull* $S$ = *UNIV*

**proposition** *aff_dim_eq_hyperplane*:
**fixes** $S$ :: *'a::euclidean_space set*
**shows** *aff_dim* $S$ = *DIM*('a) − *1* ⟷ (∃ $a$ $b$. $a \neq 0$ ∧ *affine hull* $S$ = {$x$. $a \cdot x$ = $b$})
(**is** *?lhs* = *?rhs*)

**corollary** *aff_dim_hyperplane* [*simp*]:
**fixes** $a$ :: *'a::euclidean_space*
**shows** $a \neq 0$ ⟹ *aff_dim* {$x$. $a \cdot x = r$} = *DIM*('a) − *1*

**proposition** *aff_dim_sums_Int*:
**assumes** *affine* $S$
**and** *affine* $T$
**and** $S \cap T \neq \{\}$
**shows** *aff_dim* {$x + y$| $x$ $y$. $x \in S \wedge y \in T$} = (*aff_dim* $S$ + *aff_dim* $T$) − *aff_dim*($S \cap T$)

### 5.0.5 Lower-dimensional affine subsets are nowhere dense

**proposition** *dense_complement_subspace*:
**fixes** $S$ :: *'a :: euclidean_space set*
**assumes** *dim_less*: *dim* $T$ < *dim* $S$ **and** *subspace* $S$ **shows** *closure*($S - T$) = $S$

### 5.0.6 Paracompactness

**proposition** *paracompact*:
**fixes** $S$ :: *'a* :: {*metric_space,second_countable_topology*} *set*
**assumes** $S \subseteq \bigcup \mathcal{C}$ **and** *opC*: $\bigwedge T.\ T \in \mathcal{C} \Longrightarrow$ *open* $T$
**obtains** $\mathcal{C}'$ **where** $S \subseteq \bigcup \mathcal{C}'$
**and** $\bigwedge U.\ U \in \mathcal{C}' \Longrightarrow$ *open* $U$ ∧ (∃ $T$. $T \in \mathcal{C} \wedge U \subseteq T$)
**and** $\bigwedge x.\ x \in S$
⟹ ∃ $V$. *open* $V$ ∧ $x \in V$ ∧ *finite* {$U$. $U \in \mathcal{C}' \wedge (U \cap V \neq \{\})$}

**corollary** *paracompact_closedin*:

**fixes** $S$ :: $'a$ :: $\{metric\_space,second\_countable\_topology\}$ $set$
**assumes** *cin*: $closedin\ (top\_of\_set\ U)\ S$
**and** *oin*: $\bigwedge T.\ T \in \mathcal{C} \Longrightarrow openin\ (top\_of\_set\ U)\ T$
**and** $S \subseteq \bigcup \mathcal{C}$
**obtains** $\mathcal{C}'$ **where** $S \subseteq \bigcup\ \mathcal{C}'$
**and** $\bigwedge V.\ V \in \mathcal{C}' \Longrightarrow openin\ (top\_of\_set\ U)\ V \wedge (\exists\, T.\ T \in \mathcal{C} \wedge V \subseteq T)$
**and** $\bigwedge x.\ x \in U$
$\Longrightarrow \exists\, V.\ openin\ (top\_of\_set\ U)\ V \wedge x \in V \wedge$
$finite\ \{X.\ X \in \mathcal{C}' \wedge (X \cap V \neq \{\})\}$

### 5.0.7 Covering an open set by a countable chain of compact sets

**proposition** *open_Union_compact_subsets*:
**fixes** $S$ :: $'a$::*euclidean_space set*
**assumes** *open* $S$
**obtains** $C$ **where** $\bigwedge n.\ compact(C\ n)$ $\bigwedge n.\ C\ n \subseteq S$
$\bigwedge n.\ C\ n \subseteq interior(C(Suc\ n))$
$\bigcup (range\ C) = S$
$\bigwedge K.\ [\![compact\ K;\ K \subseteq S]\!] \Longrightarrow \exists\, N.\ \forall\, n \geq N.\ K \subseteq (C\ n)$

### 5.0.8 Orthogonal complement

**definition** *orthogonal_comp* $(\_^{\perp}\ [80]\ 80)$
**where** *orthogonal_comp* $W \equiv \{x.\ \forall\, y \in W.\ orthogonal\ y\ x\}$

**proposition** *subspace_orthogonal_comp*: *subspace* $(W^{\perp})$

**proposition** *subspace_sum_orthogonal_comp*:
**fixes** $U$ :: $'a$ :: *euclidean_space set*
**assumes** *subspace* $U$
**shows** $U + U^{\perp} = UNIV$

**end**

## 5.1 Path-Connectedness

**theory** *Path_Connected*
**imports**
*Starlike*
*T1_Spaces*
**begin**

### 5.1.1 Paths and Arcs

**definition** *path* :: $(real \Rightarrow 'a$::*topological_space*$) \Rightarrow bool$

**where** $path\ g \equiv continuous\_on\ \{0..1\}\ g$

**definition** $pathstart :: (real \Rightarrow 'a::topological\_space) \Rightarrow 'a$
**where** $pathstart\ g \equiv g\ 0$

**definition** $pathfinish :: (real \Rightarrow 'a::topological\_space) \Rightarrow 'a$
**where** $pathfinish\ g \equiv g\ 1$

**definition** $path\_image :: (real \Rightarrow 'a::topological\_space) \Rightarrow 'a\ set$
**where** $path\_image\ g \equiv g\ `\ \{0\ ..\ 1\}$

**definition** $reversepath :: (real \Rightarrow 'a::topological\_space) \Rightarrow real \Rightarrow 'a$
**where** $reversepath\ g \equiv (\lambda x.\ g(1 - x))$

**definition** $joinpaths :: (real \Rightarrow 'a::topological\_space) \Rightarrow (real \Rightarrow 'a) \Rightarrow real \Rightarrow 'a$
(**infixr** $+++$ $75$)
**where** $g1\ +++\ g2 \equiv (\lambda x.\ if\ x \leq 1/2\ then\ g1\ (2 * x)\ else\ g2\ (2 * x - 1))$

**definition** $loop\_free :: (real \Rightarrow 'a::topological\_space) \Rightarrow bool$
**where** $loop\_free\ g \equiv \forall x\in\{0..1\}.\ \forall y\in\{0..1\}.\ g\ x = g\ y \longrightarrow x = y \lor x = 0 \land y = 1 \lor x = 1 \land y = 0$

**definition** $simple\_path :: (real \Rightarrow 'a::topological\_space) \Rightarrow bool$
**where** $simple\_path\ g \equiv path\ g \land loop\_free\ g$

**definition** $arc :: (real \Rightarrow 'a :: topological\_space) \Rightarrow bool$
**where** $arc\ g \equiv path\ g \land inj\_on\ g\ \{0..1\}$

### 5.1.2 Subpath

**definition** $subpath :: real \Rightarrow real \Rightarrow (real \Rightarrow 'a) \Rightarrow real \Rightarrow 'a::real\_normed\_vector$
**where** $subpath\ a\ b\ g \equiv \lambda x.\ g((b - a) * x + a)$

### 5.1.3 Shift Path to Start at Some Given Point

**definition** $shiftpath :: real \Rightarrow (real \Rightarrow 'a::topological\_space) \Rightarrow real \Rightarrow 'a$
**where** $shiftpath\ a\ f = (\lambda x.\ if\ (a + x) \leq 1\ then\ f\ (a + x)\ else\ f\ (a + x - 1))$

### 5.1.4 Straight-Line Paths

**definition** $linepath :: 'a::real\_normed\_vector \Rightarrow 'a \Rightarrow real \Rightarrow 'a$
**where** $linepath\ a\ b = (\lambda x.\ (1 - x) *_R a + x *_R b)$
**proposition** $injective\_eq\_1d\_open\_map\_UNIV$:
**fixes** $f :: real \Rightarrow real$
**assumes** $contf$: $continuous\_on\ S\ f$ **and** $S$: $is\_interval\ S$
**shows** $inj\_on\ f\ S \longleftrightarrow (\forall T.\ open\ T \land T \subseteq S \longrightarrow open(f\ `\ T))$
(**is** $?lhs = ?rhs$)

### 5.1.5 Path component

**definition** *path_component S x y* ≡
(∃ *g*. *path g* ∧ *path_image g* ⊆ *S* ∧ *pathstart g* = *x* ∧ *pathfinish g* = *y*)

**abbreviation**
*path_component_set S x* ≡ *Collect* (*path_component S x*)

### 5.1.6 Path connectedness of a space

**definition** *path_connected S* ⟷
(∀ *x*∈*S*. ∀ *y*∈*S*. ∃ *g*. *path g* ∧ *path_image g* ⊆ *S* ∧ *pathstart g* = *x* ∧ *pathfinish g* = *y*)

### 5.1.7 Path components

### 5.1.8 Paths and path-connectedness

### 5.1.9 Path components

### 5.1.10 Sphere is path-connected

**corollary** *connected_punctured_universe*:
*2* ≤ *DIM*(*'N*::*euclidean_space*) ⟹ *connected*(− {*a*::*'N*})

**proposition** *path_connected_sphere*:
**fixes** *a* :: *'a* :: *euclidean_space*
**assumes** *2* ≤ *DIM*(*'a*)
**shows** *path_connected*(*sphere a r*)

**corollary** *path_connected_complement_bounded_convex*:
**fixes** *S* :: *'a* :: *euclidean_space set*
**assumes** *bounded S convex S* **and** *2*: *2* ≤ *DIM*(*'a*)
**shows** *path_connected* (− *S*)

**proposition** *connected_open_delete*:
**assumes** *open S connected S* **and** *2*: *2* ≤ *DIM*(*'N*::*euclidean_space*)
**shows** *connected*(*S* − {*a*::*'N*})

**corollary** *path_connected_open_delete*:
**assumes** *open S connected S* **and** *2*: *2* ≤ *DIM*(*'N*::*euclidean_space*)
**shows** *path_connected*(*S* − {*a*::*'N*})

**corollary** *path_connected_punctured_ball*:

$2 \leq DIM('N::euclidean\_space) \Longrightarrow path\_connected(ball\ a\ r - \{a::'N\})$

**corollary** *connected_punctured_ball*:
$2 \leq DIM('N::euclidean\_space) \Longrightarrow connected(ball\ a\ r - \{a::'N\})$

**corollary** *connected_open_delete_finite*:
**fixes** $S\ T::'a::euclidean\_space\ set$
**assumes** $S$: $open\ S\ connected\ S$ **and** $2$: $2 \leq DIM('a)$ **and** $finite\ T$
**shows** $connected(S - T)$

### 5.1.11 Every annulus is a connected set

**proposition** *path_connected_annulus*:
**fixes** $a :: 'N::euclidean\_space$
**assumes** $2 \leq DIM('N)$
**shows** $path\_connected\ \{x.\ r1 < norm(x - a) \wedge norm(x - a) < r2\}$
$path\_connected\ \{x.\ r1 < norm(x - a) \wedge norm(x - a) \leq r2\}$
$path\_connected\ \{x.\ r1 \leq norm(x - a) \wedge norm(x - a) < r2\}$
$path\_connected\ \{x.\ r1 \leq norm(x - a) \wedge norm(x - a) \leq r2\}$

**proposition** *connected_annulus*:
**fixes** $a :: 'N::euclidean\_space$
**assumes** $2 \leq DIM('N::euclidean\_space)$
**shows** $connected\ \{x.\ r1 < norm(x - a) \wedge norm(x - a) < r2\}$
$connected\ \{x.\ r1 < norm(x - a) \wedge norm(x - a) \leq r2\}$
$connected\ \{x.\ r1 \leq norm(x - a) \wedge norm(x - a) < r2\}$
$connected\ \{x.\ r1 \leq norm(x - a) \wedge norm(x - a) \leq r2\}$

**corollary** *open_components*:
**fixes** $S :: 'a::real\_normed\_vector\ set$
**shows** $[\![open\ u;\ S \in components\ u]\!] \Longrightarrow open\ S$

**proposition** *components_open_unique*:
**fixes** $S :: 'a::real\_normed\_vector\ set$
**assumes** $pairwise\ disjnt\ A\ \bigcup A = S$
$\bigwedge X.\ X \in A \Longrightarrow open\ X \wedge connected\ X \wedge X \neq \{\}$
**shows** $components\ S = A$

### 5.1.12 The *inside* and *outside* of a Set

The inside comprises the points in a bounded connected component of the set's complement. The outside comprises the points in unbounded connected component of the complement.

**definition** *inside* **where**
$inside\ S \equiv \{x.\ (x \notin S) \wedge bounded(connected\_component\_set\ (- S)\ x)\}$

**definition** *outside* **where**
  $outside\ S \equiv -S \cap \{x.\ \neg\ bounded(connected\_component\_set\ (-\ S)\ x)\}$

### 5.1.13 Condition for an open map's image to contain a ball

**proposition** *ball_subset_open_map_image*:
  **fixes** $f :: 'a{::}heine\_borel \Rightarrow 'b :: \{real\_normed\_vector, heine\_borel\}$
  **assumes** *contf*: $continuous\_on\ (closure\ S)\ f$
    **and** *oint*: $open\ (f\ `\ interior\ S)$
    **and** *le_no*: $\bigwedge z.\ z \in frontier\ S \Longrightarrow r \leq norm(f\ z - f\ a)$
    **and** $bounded\ S\ a \in S\ 0 < r$
   **shows** $ball\ (f\ a)\ r \subseteq f\ `\ S$

**proposition** *embedding_map_into_euclideanreal*:
  **assumes** $path\_connected\_space\ X$
  **shows** $embedding\_map\ X\ euclideanreal\ f \longleftrightarrow$
      $continuous\_map\ X\ euclideanreal\ f \wedge inj\_on\ f\ (topspace\ X)$

**end**

## 5.2 Neighbourhood bases and Locally path-connected spaces

**theory** *Locally*
  **imports**
   *Path_Connected Function_Topology Sum_Topology*
**begin**

### 5.2.1 Neighbourhood Bases

### 5.2.2 Locally path-connected spaces

### 5.2.3 Locally connected spaces

### 5.2.4 Dimension of a topological space

**end**

## 5.3 Some Uncountable Sets

**theory** *Uncountable_Sets*
 **imports** *Path_Connected Continuum_Not_Denumerable*
**begin**

**end**

## 5.4 Homotopy of Maps

**theory** *Homotopy*
 **imports** *Path_Connected Product_Topology Uncountable_Sets*
**begin**

**definition** *homotopic_with*
**where**
 *homotopic_with P X Y f g* ≡
 (∃ *h*. *continuous_map* (*prod_topology* (*top_of_set* {*0*..*1*::*real*}) *X*) *Y h* ∧
 (∀ *x*. *h*(*0*, *x*) = *f x*) ∧
 (∀ *x*. *h*(*1*, *x*) = *g x*) ∧
 (∀ *t* ∈ {*0*..*1*}. *P*(λ*x*. *h*(*t*,*x*))))

**proposition** *homotopic_with*:
 **assumes** ⋀*h k*. (⋀*x*. *x* ∈ *topspace X* ⟹ *h x* = *k x*) ⟹ (*P h* ⟷ *P k*)
 **shows** *homotopic_with P X Y p q* ⟷
 (∃ *h*. *continuous_map* (*prod_topology* (*subtopology euclideanreal* {*0*..*1*}) *X*) *Y h* ∧
 (∀ *x* ∈ *topspace X*. *h*(*0*,*x*) = *p x*) ∧
 (∀ *x* ∈ *topspace X*. *h*(*1*,*x*) = *q x*) ∧
 (∀ *t* ∈ {*0*..*1*}. *P*(λ*x*. *h*(*t*, *x*))))

### 5.4.1 Homotopy with P is an equivalence relation

**proposition** *homotopic_with_trans*:
 **assumes** *homotopic_with P X Y f g* *homotopic_with P X Y g h*
 **shows** *homotopic_with P X Y f h*

### 5.4.2 Continuity lemmas

**corollary** *homotopic_compose*:
 **assumes** *homotopic_with* (λ*x*. *True*) *X Y f f′* *homotopic_with* (λ*x*. *True*) *Y Z g g′*
 **shows** *homotopic_with* (λ*x*. *True*) *X Z* (*g* ∘ *f*) (*g′* ∘ *f′*)

**proposition** *homotopic_with_compose_continuous_right*:

⟦*homotopic_with_canon* ($\lambda f.$ *p* ($f \circ h$)) *X Y f g*; *continuous_on W h*; $h \in W \to X$⟧
⟹ *homotopic_with_canon p W Y* ($f \circ h$) ($g \circ h$)

**proposition** *homotopic_with_compose_continuous_left*:
⟦*homotopic_with_canon* ($\lambda f.$ *p* ($h \circ f$)) *X Y f g*; *continuous_on Y h*; $h \in Y \to Z$⟧
⟹ *homotopic_with_canon p X Z* ($h \circ f$) ($h \circ g$)

**proposition** *homotopic_with_eq*:
**assumes** *h*: *homotopic_with P X Y f g*
**and** $f'$: $\bigwedge x.\ x \in$ *topspace X* $\Longrightarrow f'\ x = f\ x$
**and** $g'$: $\bigwedge x.\ x \in$ *topspace X* $\Longrightarrow g'\ x = g\ x$
**and** *P*: ($\bigwedge h\ k.$ ($\bigwedge x.\ x \in$ *topspace X* $\Longrightarrow h\ x = k\ x$) $\Longrightarrow P\ h \longleftrightarrow P\ k$)
**shows** *homotopic_with P X Y* $f'\ g'$

### 5.4.3 Homotopy of paths, maintaining the same endpoints

**definition** *homotopic_paths* :: ['*a set*, *real* ⇒ '*a*, *real* ⇒ '*a*::*topological_space*] ⇒ *bool*
**where**
*homotopic_paths S p q* ≡
*homotopic_with_canon* ($\lambda r.$ *pathstart r* = *pathstart p* ∧ *pathfinish r* = *pathfinish p*) {*0..1*} *S p q*

**proposition** *homotopic_paths_imp_pathstart*:
*homotopic_paths S p q* ⟹ *pathstart p* = *pathstart q*

**proposition** *homotopic_paths_imp_pathfinish*:
*homotopic_paths S p q* ⟹ *pathfinish p* = *pathfinish q*

**proposition** *homotopic_paths_refl* [*simp*]: *homotopic_paths S p p* ⟷ *path p* ∧ *path_image p* ⊆ *S*

**proposition** *homotopic_paths_sym*: *homotopic_paths S p q* ⟹ *homotopic_paths S q p*

**proposition** *homotopic_paths_sym_eq*: *homotopic_paths S p q* ⟷ *homotopic_paths S q p*

**proposition** *homotopic_paths_trans* [*trans*]:
**assumes** *homotopic_paths S p q homotopic_paths S q r*
**shows** *homotopic_paths S p r*

**proposition** *homotopic_paths_eq*:
⟦*path p*; *path_image p* ⊆ *S*; $\bigwedge t.\ t \in \{0..1\} \Longrightarrow p\ t = q\ t$⟧ ⟹ *homotopic_paths S p q*

**proposition** *homotopic_paths_reparametrize*:
**assumes** *path p*
**and** *pips*: *path_image p* $\subseteq$ *S*
**and** *contf*: *continuous_on* $\{0..1\}$ *f*
**and** *f01* :*f* $\in \{0..1\} \to \{0..1\}$
**and** [*simp*]: $f(0) = 0$ $f(1) = 1$
**and** *q*: $\bigwedge t.\ t \in \{0..1\} \Longrightarrow q(t) = p(f\ t)$
**shows** *homotopic_paths S p q*

**proposition** *homotopic_paths_reversepath*:
*homotopic_paths S* (*reversepath p*) (*reversepath q*) $\longleftrightarrow$ *homotopic_paths S p q*

**proposition** *homotopic_paths_join*:
[[*homotopic_paths S p p'*; *homotopic_paths S q q'*; *pathfinish p* = *pathstart q*]] $\Longrightarrow$ *homotopic_paths S* (*p* +++ *q*) (*p'* +++ *q'*)

**proposition** *homotopic_paths_continuous_image*:
[[*homotopic_paths S f g*; *continuous_on S h*; $h \in S \to t$]] $\Longrightarrow$ *homotopic_paths t* ($h \circ f$) ($h \circ g$)

### 5.4.4 Group properties for homotopy of paths

So taking equivalence classes under homotopy would give the fundamental group

**proposition** *homotopic_paths_rid*:
**assumes** *path p path_image p* $\subseteq$ *S*
**shows** *homotopic_paths S* (*p* +++ *linepath* (*pathfinish p*) (*pathfinish p*)) *p*

**proposition** *homotopic_paths_lid*:
[[*path p*; *path_image p* $\subseteq$ *S*]] $\Longrightarrow$ *homotopic_paths S* (*linepath* (*pathstart p*) (*pathstart p*) +++ *p*) *p*

**proposition** *homotopic_paths_assoc*:
[[*path p*; *path_image p* $\subseteq$ *S*; *path q*; *path_image q* $\subseteq$ *S*; *path r*; *path_image r* $\subseteq$ *S*; *pathfinish p* = *pathstart q*;
*pathfinish q* = *pathstart r*]]
$\Longrightarrow$ *homotopic_paths S* (*p* +++ (*q* +++ *r*)) ((*p* +++ *q*) +++ *r*)

**proposition** *homotopic_paths_rinv*:
**assumes** *path p path_image p* $\subseteq$ *S*
**shows** *homotopic_paths S* (*p* +++ *reversepath p*) (*linepath* (*pathstart p*) (*pathstart p*))

**proposition** *homotopic_paths_linv*:
**assumes** *path p path_image p* $\subseteq$ *S*

**shows** *homotopic_paths S (reversepath p +++ p) (linepath (pathfinish p) (pathfinish p))*

### 5.4.5 Homotopy of loops without requiring preservation of endpoints

**definition** *homotopic_loops* :: *'a::topological_space set* ⇒ (*real* ⇒ *'a*) ⇒ (*real* ⇒ *'a*) ⇒ *bool* **where**
*homotopic_loops S p q* ≡
*homotopic_with_canon* (*λr. pathfinish r* = *pathstart r*) {*0..1*} *S p q*

**proposition** *homotopic_loops_imp_loop*:
*homotopic_loops S p q* ⟹ *pathfinish p* = *pathstart p* ∧ *pathfinish q* = *pathstart q*

**proposition** *homotopic_loops_imp_path*:
*homotopic_loops S p q* ⟹ *path p* ∧ *path q*

**proposition** *homotopic_loops_imp_subset*:
*homotopic_loops S p q* ⟹ *path_image p* ⊆ *S* ∧ *path_image q* ⊆ *S*

**proposition** *homotopic_loops_refl*:
*homotopic_loops S p p* ⟷
*path p* ∧ *path_image p* ⊆ *S* ∧ *pathfinish p* = *pathstart p*

**proposition** *homotopic_loops_sym*: *homotopic_loops S p q* ⟹ *homotopic_loops S q p*

**proposition** *homotopic_loops_sym_eq*: *homotopic_loops S p q* ⟷ *homotopic_loops S q p*

**proposition** *homotopic_loops_trans*:
⟦*homotopic_loops S p q*; *homotopic_loops S q r*⟧ ⟹ *homotopic_loops S p r*

**proposition** *homotopic_loops_subset*:
⟦*homotopic_loops S p q*; *S* ⊆ *t*⟧ ⟹ *homotopic_loops t p q*

**proposition** *homotopic_loops_eq*:
⟦*path p*; *path_image p* ⊆ *S*; *pathfinish p* = *pathstart p*; ⋀*t. t* ∈ {*0..1*} ⟹ *p*(*t*) = *q*(*t*)⟧
⟹ *homotopic_loops S p q*

**proposition** *homotopic_loops_continuous_image*:
⟦*homotopic_loops S f g*; *continuous_on S h*; *h* ∈ *S* → *t*⟧ ⟹ *homotopic_loops t* (*h* ∘ *f*) (*h* ∘ *g*)

### 5.4.6 Relations between the two variants of homotopy

**proposition** *homotopic_paths_imp_homotopic_loops*:
  ⟦*homotopic_paths S p q*; *pathfinish p* = *pathstart p*; *pathfinish q* = *pathstart p*⟧
⟹ *homotopic_loops S p q*

**proposition** *homotopic_loops_imp_homotopic_paths_null*:
  **assumes** *homotopic_loops S p* (*linepath a a*)
    **shows** *homotopic_paths S p* (*linepath* (*pathstart p*) (*pathstart p*))

**proposition** *homotopic_loops_conjugate*:
  **fixes** *S* :: *'a*::*real_normed_vector set*
  **assumes** *path p path q* **and** *pip*: *path_image p* ⊆ *S* **and** *piq*: *path_image q* ⊆ *S*
    **and** *pq*: *pathfinish p* = *pathstart q* **and** *qloop*: *pathfinish q* = *pathstart q*
  **shows** *homotopic_loops S* (*p* +++ *q* +++ *reversepath p*) *q*

### 5.4.7 Homotopy and subpaths

**proposition** *homotopic_join_subpaths*:
  ⟦*path g*; *path_image g* ⊆ *S*; *u* ∈ {*0..1*}; *v* ∈ {*0..1*}; *w* ∈ {*0..1*}⟧
  ⟹ *homotopic_paths S* (*subpath u v g* +++ *subpath v w g*) (*subpath u w g*)

### 5.4.8 Simply connected sets

defined as "all loops are homotopic (as loops)

**definition** *simply_connected* **where**
  *simply_connected S* ≡
    ∀ *p q*. *path p* ∧ *pathfinish p* = *pathstart p* ∧ *path_image p* ⊆ *S* ∧
      *path q* ∧ *pathfinish q* = *pathstart q* ∧ *path_image q* ⊆ *S*
      ⟶ *homotopic_loops S p q*

**proposition** *simply_connected_Times*:
  **fixes** *S* :: *'a*::*real_normed_vector set* **and** *T* :: *'b*::*real_normed_vector set*
  **assumes** *S*: *simply_connected S* **and** *T*: *simply_connected T*
    **shows** *simply_connected*(*S* × *T*)

### 5.4.9 Contractible sets

**definition** *contractible* **where**
 *contractible S* ≡ ∃ *a*. *homotopic_with_canon* (λ*x*. *True*) *S S id* (λ*x*. *a*)

**proposition** *contractible_imp_simply_connected*:
  **fixes** *S* :: _::*real_normed_vector set*
  **assumes** *contractible S* **shows** *simply_connected S*

**corollary** *contractible_imp_connected*:
**fixes** $S$ :: _::*real_normed_vector set*
**shows** *contractible* $S \Longrightarrow$ *connected* $S$

### 5.4.10 Starlike sets

**definition** *starlike* $S \longleftrightarrow (\exists\, a{\in}S.\ \forall\, x{\in}S.\ closed\_segment\ a\ x \subseteq S)$

### 5.4.11 Local versions of topological properties in general

**definition** *locally* :: ($'a$::*topological_space set* $\Rightarrow$ *bool*) $\Rightarrow$ $'a$ *set* $\Rightarrow$ *bool*
**where**
*locally* $P\ S \equiv$
$\forall\, w\ x.\ openin\ (top\_of\_set\ S)\ w \wedge x \in w$
$\longrightarrow (\exists\, U\ V.\ openin\ (top\_of\_set\ S)\ U \wedge P\ V \wedge x \in U \wedge U \subseteq V \wedge V$
$\subseteq w)$

**proposition** *homeomorphism_locally_imp*:
**fixes** $S$ :: $'a$::*metric_space set* **and** $T$ :: $'b$::*t2_space set*
**assumes** $S$: *locally* $P\ S$ **and** *hom*: *homeomorphism* $S\ T\ f\ g$
**and** $Q$: $\bigwedge S\ S'.\ [\![P\ S;\ homeomorphism\ S\ S'\ f\ g]\!] \Longrightarrow Q\ S'$
**shows** *locally* $Q\ T$

### 5.4.12 An induction principle for connected sets

**proposition** *connected_induction*:
**assumes** *connected* $S$
**and** *opD*: $\bigwedge T\ a.\ [\![openin\ (top\_of\_set\ S)\ T;\ a \in T]\!] \Longrightarrow \exists\, z.\ z \in T \wedge P\ z$
**and** *opI*: $\bigwedge a.\ a \in S$
$\Longrightarrow \exists\, T.\ openin\ (top\_of\_set\ S)\ T \wedge a \in T \wedge$
$(\forall\, x \in T.\ \forall\, y \in T.\ P\ x \wedge P\ y \wedge Q\ x \longrightarrow Q\ y)$
**and** *etc*: $a \in S\ b \in S\ P\ a\ P\ b\ Q\ a$
**shows** $Q\ b$

### 5.4.13 Basic properties of local compactness

**proposition** *locally_compact*:
**fixes** $S$ :: $'a$ :: *metric_space set*
**shows**
*locally compact* $S \longleftrightarrow$
$(\forall\, x \in S.\ \exists\, u\ v.\ x \in u \wedge u \subseteq v \wedge v \subseteq S \wedge$
$openin\ (top\_of\_set\ S)\ u \wedge compact\ v)$
(**is** *?lhs* = *?rhs*)

### 5.4.14 Sura-Bura's results about compact components of sets

**proposition** *Sura_Bura_compact*:
**fixes** $S$ :: *'a::euclidean_space set*
**assumes** *compact* $S$ **and** $C$: $C \in$ *components* $S$
**shows** $C = \bigcap\{T.\ C \subseteq T \wedge$ *openin* (*top_of_set* $S$) $T \wedge$
*closedin* (*top_of_set* $S$) $T\}$
(**is** $C = \bigcap\ ?\mathcal{T}$)

**corollary** *Sura_Bura_clopen_subset*:
**fixes** $S$ :: *'a::euclidean_space set*
**assumes** $S$: *locally compact* $S$ **and** $C$: $C \in$ *components* $S$ **and** *compact* $C$
**and** $U$: *open* $U$ $C \subseteq U$
**obtains** $K$ **where** *openin* (*top_of_set* $S$) $K$ *compact* $K$ $C \subseteq K$ $K \subseteq U$

**corollary** *Sura_Bura_clopen_subset_alt*:
**fixes** $S$ :: *'a::euclidean_space set*
**assumes** $S$: *locally compact* $S$ **and** $C$: $C \in$ *components* $S$ **and** *compact* $C$
**and** *opeSU*: *openin* (*top_of_set* $S$) $U$ **and** $C \subseteq U$
**obtains** $K$ **where** *openin* (*top_of_set* $S$) $K$ *compact* $K$ $C \subseteq K$ $K \subseteq U$

**corollary** *Sura_Bura*:
**fixes** $S$ :: *'a::euclidean_space set*
**assumes** *locally compact* $S$ $C \in$ *components* $S$ *compact* $C$
**shows** $C = \bigcap\ \{K.\ C \subseteq K \wedge$ *compact* $K \wedge$ *openin* (*top_of_set* $S$) $K\}$
(**is** $C = ?rhs$)

### 5.4.15 Special cases of local connectedness and path connectedness

**proposition** *locally_path_connected*:
*locally path_connected* $S \longleftrightarrow$
($\forall\ V\ x.$ *openin* (*top_of_set* $S$) $V \wedge x \in V$
$\longrightarrow$ ($\exists\ U.$ *openin* (*top_of_set* $S$) $U \wedge$ *path_connected* $U \wedge x \in U \wedge U \subseteq$
$V$))

**proposition** *locally_path_connected_open_path_component*:
*locally path_connected* $S \longleftrightarrow$
($\forall\ t\ x.$ *openin* (*top_of_set* $S$) $t \wedge x \in t$
$\longrightarrow$ *openin* (*top_of_set* $S$) (*path_component_set* $t$ $x$))

**proposition** *locally_connected_im_kleinen*:
*locally connected* $S \longleftrightarrow$
($\forall\ v\ x.$ *openin* (*top_of_set* $S$) $v \wedge x \in v$
$\longrightarrow$ ($\exists\ u.$ *openin* (*top_of_set* $S$) $u \wedge$
$x \in u \wedge u \subseteq v \wedge$

$(\forall y.\ y \in u \longrightarrow (\exists c.\ connected\ c \wedge c \subseteq v \wedge x \in c \wedge y \in c))))$
(**is** $?lhs = ?rhs$)

**proposition** *locally_path_connected_im_kleinen*:
$locally\ path\_connected\ S \longleftrightarrow$
$(\forall v\ x.\ openin\ (top\_of\_set\ S)\ v \wedge x \in v$
$\longrightarrow (\exists u.\ openin\ (top\_of\_set\ S)\ u \wedge$
$x \in u \wedge u \subseteq v \wedge$
$(\forall y.\ y \in u \longrightarrow (\exists p.\ path\ p \wedge path\_image\ p \subseteq v \wedge$
$pathstart\ p = x \wedge pathfinish\ p = y))))$
(**is** $?lhs = ?rhs$)

### 5.4.16 Relations between components and path components

**proposition** *locally_connected_quotient_image*:
**assumes** *lcS*: $locally\ connected\ S$
**and** *oo*: $\bigwedge T.\ T \subseteq f\ `\ S$
$\Longrightarrow openin\ (top\_of\_set\ S)\ (S \cap f\ -`\ T) \longleftrightarrow$
$openin\ (top\_of\_set\ (f\ `\ S))\ T$
**shows** $locally\ connected\ (f\ `\ S)$
**proposition** *locally_path_connected_quotient_image*:
**assumes** *lcS*: $locally\ path\_connected\ S$
**and** *oo*: $\bigwedge T.\ T \subseteq f\ `\ S$
$\Longrightarrow openin\ (top\_of\_set\ S)\ (S \cap f\ -`\ T) \longleftrightarrow openin\ (top\_of\_set\ (f$
$`\ S))\ T$
**shows** $locally\ path\_connected\ (f\ `\ S)$

### 5.4.17 Existence of isometry between subspaces of same dimension

**proposition** *isometries_subspaces*:
**fixes** $S :: 'a::euclidean\_space\ set$
**and** $T :: 'b::euclidean\_space\ set$
**assumes** *S*: $subspace\ S$
**and** *T*: $subspace\ T$
**and** *d*: $dim\ S = dim\ T$
**obtains** $f\ g$ **where** $linear\ f\ linear\ g\ f\ `\ S = T\ g\ `\ T = S$
$\bigwedge x.\ x \in S \Longrightarrow norm(f\ x) = norm\ x$
$\bigwedge x.\ x \in T \Longrightarrow norm(g\ x) = norm\ x$
$\bigwedge x.\ x \in S \Longrightarrow g(f\ x) = x$
$\bigwedge x.\ x \in T \Longrightarrow f(g\ x) = x$

**corollary** *isometry_subspaces*:
**fixes** $S :: 'a::euclidean\_space\ set$
**and** $T :: 'b::euclidean\_space\ set$

**assumes** $S$: *subspace* $S$
**and** $T$: *subspace* $T$
**and** $d$: $dim\ S = dim\ T$
**obtains** $f$ **where** $linear\ f\ f\ `\ S = T\ \bigwedge x.\ x \in S \Longrightarrow norm(f\ x) = norm\ x$

**corollary** *isomorphisms_UNIV_UNIV*:
**assumes** $DIM('M) = DIM('N)$
**obtains** $f::'M::euclidean\_space \Rightarrow 'N::euclidean\_space$ **and** $g$
**where** $linear\ f\ linear\ g$
$\bigwedge x.\ norm(f\ x) = norm\ x\ \bigwedge y.\ norm(g\ y) = norm\ y$
$\bigwedge x.\ g\ (f\ x) = x\ \bigwedge y.\ f(g\ y) = y$

### 5.4.18 Retracts, in a general sense, preserve (co)homotopic triviality)

**locale** *Retracts* =
**fixes** $S\ h\ t\ k$
**assumes** *conth*: *continuous_on* $S\ h$
**and** *imh*: $h\ `\ S = t$
**and** *contk*: *continuous_on* $t\ k$
**and** *imk*: $k \in t \rightarrow S$
**and** *idhk*: $\bigwedge y.\ y \in t \Longrightarrow h(k\ y) = y$

**begin**

### 5.4.19 Homotopy equivalence

### 5.4.20 Homotopy equivalence of topological spaces.

**definition** *homotopy_equivalent_space*
(**infix** *homotopy'_equivalent'_space 50*)
**where** $X$ *homotopy_equivalent_space* $Y \equiv$
($\exists f\ g.$ *continuous_map* $X\ Y\ f\ \wedge$
*continuous_map* $Y\ X\ g\ \wedge$
*homotopic_with* $(\lambda x.\ True)\ X\ X\ (g \circ f)\ id\ \wedge$
*homotopic_with* $(\lambda x.\ True)\ Y\ Y\ (f \circ g)\ id)$

### 5.4.21 Contractible spaces

**corollary** *contractible_space_euclideanreal*: *contractible_space euclideanreal*

**abbreviation** *homotopy_eqv* :: *'a::topological_space set* ⇒ *'b::topological_space set* ⇒ *bool*
(**infix** *homotopy'_eqv 50*)
**where** *S homotopy_eqv T* ≡ *top_of_set S homotopy_equivalent_space top_of_set T*

**corollary** *bounded_path_connected_Compl_real*:
**fixes** *S* :: *real set*
**assumes** *bounded S path_connected*(− *S*) **shows** *S* = {}

**proposition** *path_connected_convex_diff_countable*:
**fixes** *U* :: *'a::euclidean_space set*
**assumes** *convex U* ¬ *collinear U countable S*
**shows** *path_connected*(*U* − *S*)

**corollary** *connected_convex_diff_countable*:
**fixes** *U* :: *'a::euclidean_space set*
**assumes** *convex U* ¬ *collinear U countable S*
**shows** *connected*(*U* − *S*)

**proposition** *path_connected_openin_diff_countable*:
**fixes** *S* :: *'a::euclidean_space set*
**assumes** *connected S* **and** *ope*: *openin* (*top_of_set* (*affine hull S*)) *S*
**and** ¬ *collinear S countable T*
**shows** *path_connected*(*S* − *T*)

**corollary** *connected_openin_diff_countable*:
**fixes** *S* :: *'a::euclidean_space set*
**assumes** *connected S* **and** *ope*: *openin* (*top_of_set* (*affine hull S*)) *S*
**and** ¬ *collinear S countable T*
**shows** *connected*(*S* − *T*)

**corollary** *path_connected_open_diff_countable*:
**fixes** *S* :: *'a::euclidean_space set*
**assumes** *2* ≤ *DIM*(*'a*) *open S connected S countable T*
**shows** *path_connected*(*S* − *T*)

**corollary** *connected_open_diff_countable*:
**fixes** *S* :: *'a::euclidean_space set*
**assumes** *2* ≤ *DIM*(*'a*) *open S connected S countable T*
**shows** *connected*(*S* − *T*)

### 5.4.22 Nullhomotopic mappings

**proposition** *nullhomotopic_from_sphere_extension*:
**fixes** $f$ :: *'M::euclidean_space* ⇒ *'a::real_normed_vector*
**shows** (∃ *c*. *homotopic_with_canon* (λ*x*. *True*) (*sphere a r*) *S f* (λ*x*. *c*)) ⟷
(∃ *g*. *continuous_on* (*cball a r*) *g* ∧ *g* ‘ (*cball a r*) ⊆ *S* ∧
(∀ *x* ∈ *sphere a r*. *g x* = *f x*))
(**is** *?lhs* = *?rhs*)

**end**

## 5.5 Euclidean space and n-spheres, as subtopologies of n-dimensional space

**theory** *Abstract_Euclidean_Space*
**imports** *Homotopy Locally*
**begin**

### 5.5.1 Euclidean spaces as abstract topologies

### 5.5.2 n-dimensional spheres

**proposition** *contractible_space_upper_hemisphere*:
**assumes** $k \leq n$
**shows** *contractible_space*(*subtopology* (*nsphere n*) {*x*. *x k* ≥ *0*})

**corollary** *contractible_space_lower_hemisphere*:
**assumes** $k \leq n$
**shows** *contractible_space*(*subtopology* (*nsphere n*) {*x*. *x k* ≤ *0*})

**proposition** *nullhomotopic_nonsurjective_sphere_map*:
**assumes** *f*: *continuous_map* (*nsphere p*) (*nsphere p*) *f*
**and** *fim*: *f* ‘ (*topspace*(*nsphere p*)) ≠ *topspace*(*nsphere p*)
**obtains** *a* **where** *homotopic_with* (λ*x*. *True*) (*nsphere p*) (*nsphere p*) *f* (λ*x*. *a*)

**end**

## 5.6 Various Forms of Topological Spaces

**theory** *Abstract_Topological_Spaces*
**imports** *Lindelof_Spaces Locally Abstract_Euclidean_Space Sum_Topology FSigma*
**begin**

### 5.6.1 Connected topological spaces

### 5.6.2 The notion of "separated between" (complement of "connected between)"

### 5.6.3 Connected components

### 5.6.4 Monotone maps (in the general topological sense)

**proposition** *connected_space_monotone_quotient_map_preimage*:
**assumes** *f*: *monotone_map X Y f quotient_map X Y f* **and** *connected_space Y*
**shows** *connected_space X*

### 5.6.5 Other countability properties

### 5.6.6 Neigbourhood bases EXTRAS

### 5.6.7 $T_0$ spaces and the Kolmogorov quotient

**proposition** *t0_space_product_topology*:
*t0_space* (*product_topology X I*) ⟷ *product_topology X I* = *trivial_topology* ∨ (∀ *i* ∈ *I*. *t0_space* (*X i*))
(**is** *?lhs=?rhs*)

### 5.6.8 Kolmogorov quotients

### 5.6.9 Closed diagonals and graphs

### 5.6.10 KC spaces, those where all compact sets are closed.

**proposition** *kc_space_prod_topology_left*:
**assumes** *X*: *kc_space X* **and** *Y*: *Hausdorff_space Y*
**shows** *kc_space* (*prod_topology X Y*)

### 5.6.11 Technical results about proper maps, perfect maps, etc

### 5.6.12 Regular spaces

**proposition** *regular_space_continuous_proper_map_image*:
**assumes** *regular_space X* **and** *contf*: *continuous_map X Y f* **and** *pmapf*: *proper_map X Y f*
**and** *fim*: *f* ‘ (*topspace X*) = *topspace Y*
**shows** *regular_space Y*

**proposition** *regular_space_perfect_map_image_eq*:
**assumes** *Hausdorff_space X* **and** *perf*: *perfect_map X Y f*
**shows** *regular_space X* ⟷ *regular_space Y* (**is** *?lhs*=*?rhs*)

### 5.6.13 Locally compact spaces

**proposition** *quotient_map_prod_right*:
**assumes** *loc*: *locally_compact_space Z*
**and** *reg*: *Hausdorff_space Z* ∨ *regular_space Z*
**and** *f*: *quotient_map X Y f*
**shows** *quotient_map* (*prod_topology Z X*) (*prod_topology Z Y*) (λ(*x*,*y*). (*x*,*f y*))

### 5.6.14 Special characterizations of classes of functions into and out of R

### 5.6.15 Normal spaces

### 5.6.16 Hereditary topological properties

### 5.6.17 Limits in a topological space

### 5.6.18 Quasi-components

### 5.6.19 Additional quasicomponent and continuum properties like Boundary Bumping

### 5.6.20 Compactly generated spaces (k-spaces)

**end**

## 5.7 Abstract Metric Spaces

**theory** *Abstract_Metric_Spaces*
 **imports** *Elementary_Metric_Spaces Abstract_Limits Abstract_Topological_Spaces*
**begin**

### 5.7.1 Metric topology

### 5.7.2 Bounded sets

### 5.7.3 Subspace of a metric space

### 5.7.4 Abstract type of metric spaces

### 5.7.5 The discrete metric

### 5.7.6 Metrizable spaces

### 5.7.7 Limits at a point in a topological space

### 5.7.8 Normal spaces and metric spaces

### 5.7.9 Topological limitin in metric spaces

### 5.7.10 Cauchy sequences and complete metric spaces

### 5.7.11 Totally bounded subsets of metric spaces

### 5.7.12 Compactness in metric spaces

### 5.7.13 Continuous functions on metric spaces

#### 5.7.14 Completely metrizable spaces

#### 5.7.15 Product metric

#### 5.7.16 The "atin-within" filter for topologies

#### 5.7.17 More sequential characterizations in a metric space

#### 5.7.18 Three strong notions of continuity for metric spaces

#### 5.7.19 Isometries

#### 5.7.20 "Capped" equivalent bounded metrics and general product metrics

**proposition** *metrizable_space_product_topology*:
*metrizable_space* (*product_topology* $X$ $I$) $\longleftrightarrow$
(*product_topology* $X$ $I$) = *trivial_topology* $\vee$
*countable* $\{i \in I.\ \neg\ (\exists\, a.\ topspace(X\ i) \subseteq \{a\})\}$ $\wedge$
($\forall\, i \in I.$ *metrizable_space* ($X$ $i$))

**proposition** *completely_metrizable_space_product_topology*:
  *completely_metrizable_space* (*product_topology X I*) $\longleftrightarrow$
    (*product_topology X I*) = *trivial_topology* $\vee$
    *countable* $\{i \in I.\ \neg\ (\exists\, a.\ topspace(X\ i) \subseteq \{a\})\}$ $\wedge$
    $(\forall\, i \in I.$ *completely_metrizable_space* $(X\ i))$

**end**

## 5.8 Infinite sums

**theory** *Infinite_Sum*
  **imports**
    *Elementary_Topology*
    *HOL−Library.Extended_Nonnegative_Real*
    *HOL−Library.Complex_Order*
**begin**

### 5.8.1 Definition and syntax

### 5.8.2 General properties

### 5.8.3 Absolute convergence

### 5.8.4 Extended reals and nats

### 5.8.5 Real numbers

### 5.8.6 Complex numbers

**class** *complete_uniform_space* = *uniform_space* +
**assumes** *cauchy_filter_convergent'*: *cauchy_filter* $(F :: 'a\ filter) \Longrightarrow F \neq bot \Longrightarrow$ *convergent_filter F*

**theorem** (**in** *uniform_space*) *controlled_sequences_convergent_imp_complete*:
**fixes** $U :: nat \Rightarrow ('a \times 'a)\ set$
**assumes** *gen*: *countably_generated_filter* (*uniformity* :: $('a \times 'a)$ *filter*)
**assumes** *U*: $\bigwedge n.$ *eventually* $(\lambda z.\ z \in U\ n)$ *uniformity*
**assumes** *conv*: $\bigwedge(u :: nat \Rightarrow 'a).\ (\bigwedge N\ m\ n.\ N \leq m \Longrightarrow N \leq n \Longrightarrow (u\ m,\ u\ n) \in U\ N) \Longrightarrow$ *convergent u*
**shows** *class.complete_uniform_space open uniformity*

**theorem** (**in** *uniform_space*) *controlled_seq_imp_Cauchy_seq*:
**fixes** $U :: nat \Rightarrow ('a \times 'a)\ set$
**assumes** *U*: $\bigwedge P.$ *eventually P uniformity* $\Longrightarrow (\exists\, n.\ \forall\, x{\in}U\ n.\ P\ x)$
**assumes** *controlled*: $\bigwedge N\ m\ n.\ N \leq m \Longrightarrow N \leq n \Longrightarrow (f\ m,\ f\ n) \in U\ N$
**shows** *Cauchy f*

**theorem** (**in** *uniform_space*) *Cauchy_seq_convergent_imp_complete*:
**fixes** $U :: nat \Rightarrow ('a \times 'a)\ set$
**assumes** *gen*: *countably_generated_filter* (*uniformity* :: $('a \times 'a)$ *filter*)
**assumes** *conv*: $\bigwedge(u :: nat \Rightarrow 'a).$ *Cauchy u* $\Longrightarrow$ *convergent u*
**shows** *class.complete_uniform_space open uniformity*

**end**

## 5.9 Ordered Euclidean Space

**theory** *Ordered_Euclidean_Space*
**imports**
*Convex_Euclidean_Space Abstract_Limits*
*HOL−Library.Product_Order*
**beginclass** *ordered_euclidean_space* = *ord* + *inf* + *sup* + *abs* + *Inf* + *Sup* + *euclidean_space* +
**assumes** *eucl_le*: $x \leq y \longleftrightarrow (\forall\, i{\in}Basis.\ x \cdot i \leq y \cdot i)$
**assumes** *eucl_less_le_not_le*: $x < y \longleftrightarrow x \leq y \wedge \neg\ y \leq x$
**assumes** *eucl_inf*: $inf\ x\ y = (\sum i{\in}Basis.\ inf\ (x \cdot i)\ (y \cdot i) *_R i)$

**assumes** *eucl_sup*: $sup\ x\ y = (\sum i{\in}Basis.\ sup\ (x \cdot i)\ (y \cdot i) *_R i)$
**assumes** *eucl_Inf*: $Inf\ X = (\sum i{\in}Basis.\ (INF\ x{\in}X.\ x \cdot i) *_R i)$
**assumes** *eucl_Sup*: $Sup\ X = (\sum i{\in}Basis.\ (SUP\ x{\in}X.\ x \cdot i) *_R i)$
**assumes** *eucl_abs*: $|x| = (\sum i{\in}Basis.\ |x \cdot i| *_R i)$
**begin**

**proposition** *compact_attains_Inf_componentwise*:
**fixes** $b$::*'a::ordered_euclidean_space*
**assumes** $b \in Basis$ **assumes** $X \neq \{\}$ *compact X*
**obtains** $x$ **where** $x \in X$ $x \cdot b = Inf\ X \cdot b$ $\bigwedge y.\ y \in X \Longrightarrow x \cdot b \leq y \cdot b$

**proposition**
*compact_attains_Sup_componentwise*:
**fixes** $b$::*'a::ordered_euclidean_space*
**assumes** $b \in Basis$ **assumes** $X \neq \{\}$ *compact X*
**obtains** $x$ **where** $x \in X$ $x \cdot b = Sup\ X \cdot b$ $\bigwedge y.\ y \in X \Longrightarrow y \cdot b \leq x \cdot b$
**proposition**
**fixes** $a$ :: *'a::ordered_euclidean_space*
**shows** *cbox_interval*: $cbox\ a\ b = \{a..b\}$
**and** *interval_cbox*: $\{a..b\} = cbox\ a\ b$
**and** *eucl_le_atMost*: $\{x.\ \forall i{\in}Basis.\ x \cdot i <= a \cdot i\} = \{..a\}$
**and** *eucl_le_atLeast*: $\{x.\ \forall i{\in}Basis.\ a \cdot i <= x \cdot i\} = \{a..\}$

**instantiation** *vec* :: (*ordered_euclidean_space*, *finite*) *ordered_euclidean_space*
**begin**

**definition** $inf\ x\ y = (\chi\ i.\ inf\ (x\ \$\ i)\ (y\ \$\ i))$
**definition** $sup\ x\ y = (\chi\ i.\ sup\ (x\ \$\ i)\ (y\ \$\ i))$
**definition** $Inf\ X = (\chi\ i.\ (INF\ x{\in}X.\ x\ \$\ i))$
**definition** $Sup\ X = (\chi\ i.\ (SUP\ x{\in}X.\ x\ \$\ i))$
**definition** $|x| = (\chi\ i.\ |x\ \$\ i|)$

**end**

## 5.10 Arcwise-Connected Sets

**theory** *Arcwise_Connected*
**imports** *Path_Connected Ordered_Euclidean_Space HOL−Computational_Algebra.Primes*
**begin**

### 5.10.1 The Brouwer reduction theorem

**theorem** *Brouwer_reduction_theorem_gen*:
**fixes** $S$ :: *'a::euclidean_space set*
**assumes** *closed S* $\varphi$ $S$
**and** $\varphi$: $\bigwedge F.\ [\![\bigwedge n.\ closed(F\ n);\ \bigwedge n.\ \varphi(F\ n);\ \bigwedge n.\ F(Suc\ n) \subseteq F\ n]\!] \Longrightarrow$
$\varphi(\bigcap(range\ F))$
**obtains** $T$ **where** $T \subseteq S$ *closed T* $\varphi$ $T$ $\bigwedge U.\ [\![U \subseteq S;\ closed\ U;\ \varphi\ U]\!] \Longrightarrow \neg\ (U$

$\subset T)$

**corollary** *Brouwer_reduction_theorem*:
**fixes** $S$ :: *'a::euclidean_space set*
**assumes** *compact* $S$ $\varphi$ $S$ $S \neq \{\}$
**and** $\varphi$: $\bigwedge F.\ [\![\bigwedge n.\ compact(F\ n);\ \bigwedge n.\ F\ n \neq \{\};\ \bigwedge n.\ \varphi(F\ n);\ \bigwedge n.\ F(Suc\ n) \subseteq F\ n]\!] \Longrightarrow \varphi(\bigcap(range\ F))$
**obtains** $T$ **where** $T \subseteq S$ *compact* $T$ $T \neq \{\}$ $\varphi$ $T$
$\bigwedge U.\ [\![U \subseteq S;\ closed\ U;\ U \neq \{\};\ \varphi\ U]\!] \Longrightarrow \neg\ (U \subset T)$

### 5.10.2 Density of points with dyadic rational coordinates

**proposition** *closure_dyadic_rationals*:
$closure\ (\bigcup k.\ \bigcup f \in Basis \to \mathbb{Z}.\ \{\ \sum i :: 'a :: euclidean\_space \in Basis.\ (f\ i\ /\ 2\hat{\ }k) *_R\ i\ \}) = UNIV$

**corollary** *closure_rational_coordinates*:
$closure\ (\bigcup f \in Basis \to \mathbb{Q}.\ \{\ \sum i :: 'a :: euclidean\_space \in Basis.\ f\ i *_R\ i\ \}) = UNIV$

**theorem** *homeomorphic_monotone_image_interval*:
**fixes** $f$ :: *real* $\Rightarrow$ *'a::{real_normed_vector,complete_space}*
**assumes** *cont_f*: *continuous_on* $\{0..1\}$ $f$
**and** *conn*: $\bigwedge y.\ connected\ (\{0..1\} \cap f -\text{`}\ \{y\})$
**and** *f_1not0*: $f\ 1 \neq f\ 0$
**shows** $(f\ \text{`}\ \{0..1\})$ *homeomorphic* $\{0..1::real\}$

**theorem** *path_contains_arc*:
**fixes** $p$ :: *real* $\Rightarrow$ *'a::{complete_space,real_normed_vector}*
**assumes** *path* $p$ **and** $a$: *pathstart* $p = a$ **and** $b$: *pathfinish* $p = b$ **and** $a \neq b$
**obtains** $q$ **where** *arc* $q$ *path_image* $q \subseteq$ *path_image* $p$ *pathstart* $q = a$ *pathfinish* $q = b$

**corollary** *path_connected_arcwise*:
**fixes** $S$ :: *'a::{complete_space,real_normed_vector} set*
**shows** *path_connected* $S \longleftrightarrow$
$(\forall x \in S.\ \forall y \in S.\ x \neq y \longrightarrow (\exists g.\ arc\ g \wedge path\_image\ g \subseteq S \wedge pathstart\ g = x \wedge pathfinish\ g = y))$
(**is** *?lhs = ?rhs*)

**corollary** *arc_connected_trans*:
  **fixes** $g$ :: *real* ⇒ ′*a*::{*complete_space*,*real_normed_vector*}
  **assumes** *arc g arc h pathfinish g* = *pathstart h pathstart g* ≠ *pathfinish h*
  **obtains** *i* **where** *arc i path_image i* ⊆ *path_image g* ∪ *path_image h*
    *pathstart i* = *pathstart g pathfinish i* = *pathfinish h*

### 5.10.3 Accessibility of frontier points

**end**

## 5.11 The Urysohn lemma, its consequences and other advanced material about metric spaces

**theory** *Urysohn*
**imports** *Abstract_Topological_Spaces Abstract_Metric_Spaces Infinite_Sum Arcwise_Connected*
**begin**

### 5.11.1 Urysohn lemma and Tietze's theorem

**proposition** *Urysohn_lemma*:
  **fixes** *a b* :: *real*
  **assumes** *normal_space X closedin X S closedin X T disjnt S T a* ≤ *b*
  **obtains** *f* **where** *continuous_map X* (*top_of_set* {*a*..*b*}) *f f* ‘ *S* ⊆ {*a*} *f* ‘ *T* ⊆ {*b*}

**theorem** *Tietze_extension_closed_real_interval*:
  **assumes** *normal_space X* **and** *closedin X S*
    **and** *contf*: *continuous_map* (*subtopology X S*) *euclideanreal f*
    **and** *fim*: *f* ‘ *S* ⊆ {*a*..*b*} **and** *a* ≤ *b*
  **obtains** *g*
  **where** *continuous_map X euclideanreal g*
    ⋀*x*. *x* ∈ *S* ⟹ *g x* = *f x g* ‘ *topspace X* ⊆ {*a*..*b*}

**theorem** *Tietze_extension_realinterval*:
  **assumes** *XS*: *normal_space X closedin X S* **and** *T*: *is_interval T T* ≠ {}
    **and** *contf*: *continuous_map* (*subtopology X S*) *euclideanreal f*
    **and** *f* ‘ *S* ⊆ *T*
  **obtains** *g* **where** *continuous_map X euclideanreal g g* ‘ *topspace X* ⊆ *T* ⋀*x*. *x* ∈ *S* ⟹ *g x* = *f x*

### 5.11.2 Random metric space stuff

### 5.11.3 Hereditarily normal spaces

### 5.11.4 Completely regular spaces

**proposition** *locally_compact_regular_imp_completely_regular_space*:
  **assumes** *locally_compact_space X Hausdorff_space X* ∨ *regular_space X*
  **shows** *completely_regular_space X*

**proposition** *completely_regular_space_product_topology*:
  *completely_regular_space* (*product_topology X I*) ⟷
  (∃ *i*∈*I*. *X i* = *trivial_topology*) ∨ (∀ *i* ∈ *I*. *completely_regular_space* (*X i*))
  (**is** *?lhs* ⟷ *?rhs*)

### 5.11.5 More generally, the k-ification functor

### 5.11.6 One-point compactifications and the Alexandroff extension construction

**proposition** *kc_space_one_point_compactification_gen*:
  **assumes** *compact_space X*
  **shows** *kc_space X* ⟷
    *openin X* (*topspace X* − {*a*}) ∧ (∀ *K*. *compactin X K* ∧ *a*∉*K* ⟶ *closedin X K*) ∧
    *k_space* (*subtopology X* (*topspace X* − {*a*})) ∧ *kc_space* (*subtopology X* (*topspace X* − {*a*}))
  (**is** *?lhs* ⟷ *?rhs*)

**proposition** *istopology_Alexandroff_open*: *istopology* (*Alexandroff_open X*)

**proposition** *regular_space_one_point_compactification*:
**assumes** *compact_space X* **and** *ope*: *openin X* (*topspace X* − {*a*})
**and** §: ⋀*K*. ⟦*compactin* (*subtopology X* (*topspace X* − {*a*})) *K*; *closedin* (*subtopology X* (*topspace X* − {*a*})) *K*⟧ ⟹ *closedin X K*
**shows** *regular_space X* ⟷
*regular_space* (*subtopology X* (*topspace X* − {*a*})) ∧ *locally_compact_space* (*subtopology X* (*topspace X* − {*a*}))
(**is** *?lhs* ⟷ *?rhs*)

**proposition** *Hausdorff_space_one_point_compactification_asymmetric_prod*:
**assumes** *compact_space X*
**shows** *Hausdorff_space X* ⟷
*kc_space* (*prod_topology X* (*subtopology X* (*topspace X* − {*a*}))) ∧
*k_space* (*prod_topology X* (*subtopology X* (*topspace X* − {*a*}))) (**is** *?lhs* ⟷ *?rhs*)

### 5.11.7 Extending continuous maps "pointwise" in a regular space

### 5.11.8 Extending Cauchy continuous functions to the closure

### 5.11.9 Metric space of bounded functions

#### 5.11.10 Metric space of continuous bounded functions

#### 5.11.11 Existence of completion for any metric space M as a subspace of M=>R

#### 5.11.12 Contractions

#### 5.11.13 The Baire Category Theorem

#### 5.11.14 Sierpinski-Hausdorff type results about countable closed unions

#### 5.11.15 The Tychonoff embedding

#### 5.11.16 Urysohn and Tietze analogs for completely regular spaces

#### 5.11.17 Size bounds on connected or path-connected spaces

#### 5.11.18 Lavrentiev extension etc

### 5.11.19 Embedding in products and hence more about completely metrizable spaces

### 5.11.20 Theorems from Kuratowski

### 5.11.21 A perfect set in common cases must have at least the cardinality of the continuum

**proposition** *Kuratowski_component_number_invariance_aux*:
**assumes** *compact_space X* **and** *HsX*: *Hausdorff_space X*
**and** *lcX*: *locally_connected_space X* **and** *hnX*: *hereditarily normal_space X*
**and** *hom*: (*subtopology X S*) *homeomorphic_space* (*subtopology X T*)
**and** *leXS*: {..<*n*::*nat*} ≲ *connected_components_of* (*subtopology X* (*topspace X* − *S*))
**assumes** §: ⋀*S T*.
⟦*closedin X S*; *closedin X T*; (*subtopology X S*) *homeomorphic_space* (*subtopology X T*);
{..<*n*::*nat*} ≲ *connected_components_of* (*subtopology X* (*topspace X* − *S*))⟧
⟹ {..<*n*::*nat*} ≲ *connected_components_of* (*subtopology X* (*topspace X* − *T*))
**shows** {..<*n*::*nat*} ≲ *connected_components_of* (*subtopology X* (*topspace X* − *T*))

**theorem** *Kuratowski_component_number_invariance*:
**assumes** *compact_space X Hausdorff_space X locally_connected_space X hereditarily normal_space X*
**shows** ((∀ *S T n*.
*closedin X S* ∧ *closedin X T* ∧
(*subtopology X S*) *homeomorphic_space* (*subtopology X T*)
⟶ (*connected_components_of*
(*subtopology X* (*topspace X* − *S*)) ≈ {..<*n*::*nat*} ⟷
*connected_components_of*
(*subtopology X* (*topspace X* − *T*)) ≈ {..<*n*::*nat*})) ⟷
(∀ *S T n*.
(*subtopology X S*) *homeomorphic_space* (*subtopology X T*)

```
      ⟶ (connected_components_of
            (subtopology X (topspace X − S)) ≈ {..<n::nat} ⟷
           connected_components_of
            (subtopology X (topspace X − T)) ≈ {..<n::nat})))
    (is ?lhs = ?rhs)
```

**end**
**theory** *Isolated*
  **imports** *HOL−Analysis.Elementary_Metric_Spaces*

**begin**

### 5.11.22 Isolate and discrete

**end**

## 5.12 Operator Norm

**theory** *Operator_Norm*
**imports** *Complex_Main*
**begin**

**definition**
*onorm* :: (*'a*::*real_normed_vector* ⇒ *'b*::*real_normed_vector*) ⇒ *real* **where**
*onorm f* = (*SUP x. norm* (*f x*) / *norm x*)

**proposition** *onorm_bound*:
  **assumes** *0* ≤ *b* **and** ⋀*x. norm* (*f x*) ≤ *b* ∗ *norm x*
  **shows** *onorm f* ≤ *b*

**end**

## 5.13 Limits on the Extended Real Number Line

**theory** *Extended_Real_Limits*
**imports**
  *Topology_Euclidean_Space*
  *HOL−Library.Extended_Real*
  *HOL−Library.Extended_Nonnegative_Real*
  *HOL−Library.Indicator_Function*
**begin**

### 5.13.1 Extended-Real.thy

**Continuity of addition**

**Continuity of multiplication**

**Continuity of division**

### 5.13.2 Extended-Nonnegative-Real.thy

### 5.13.3 monoset

### 5.13.4 Relate extended reals and the indicator function

**end**

## 5.14 Radius of Convergence and Summation Tests

**theory** *Summation_Tests*
**imports**
  *Complex_Main*
  *HOL−Library.Discrete*
  *HOL−Library.Extended_Real*
  *HOL−Library.Liminf_Limsup*
  *Extended_Real_Limits*
**begin**

### 5.14.1 Convergence tests for infinite sums

**theorem** *root_test_convergence'*:
  **fixes** $f$ :: *nat* ⇒ *'a* :: *banach*
  **defines** $l \equiv limsup\ (\lambda n.\ ereal\ (root\ n\ (norm\ (f\ n))))$
  **assumes** $l$: $l < 1$
  **shows** *summable f*

**theorem** *root_test_divergence*:
  **fixes** $f$ :: *nat* ⇒ *'a* :: *banach*
  **defines** $l \equiv limsup\ (\lambda n.\ ereal\ (root\ n\ (norm\ (f\ n))))$
  **assumes** $l$: $l > 1$
  **shows** ¬*summable f*

**theorem** *condensation_test*:
  **assumes** *mono*: $\bigwedge m.\ 0 < m \Longrightarrow f\ (Suc\ m) \leq f\ m$

**assumes** *nonneg*: $\bigwedge n.\ f\ n \geq 0$
**shows** *summable f* $\longleftrightarrow$ *summable* $(\lambda n.\ 2\hat{\ }n * f\ (2\hat{\ }n))$

**theorem** *summable_complex_powr_iff*:
**assumes** $Re\ s < -1$
**shows** *summable* $(\lambda n.\ exp\ (of\_real\ (ln\ (of\_nat\ n)) * s))$
**theorem** *kummers_test_convergence*:
**fixes** $f\ p :: nat \Rightarrow real$
**assumes** *pos_f*: *eventually* $(\lambda n.\ f\ n > 0)$ *sequentially*
**assumes** *nonneg_p*: *eventually* $(\lambda n.\ p\ n \geq 0)$ *sequentially*
**defines** $l \equiv liminf\ (\lambda n.\ ereal\ (p\ n * f\ n\ /\ f\ (Suc\ n) - p\ (Suc\ n)))$
**assumes** *l*: $l > 0$
**shows** *summable f*

**theorem** *kummers_test_divergence*:
**fixes** $f\ p :: nat \Rightarrow real$
**assumes** *pos_f*: *eventually* $(\lambda n.\ f\ n > 0)$ *sequentially*
**assumes** *pos_p*: *eventually* $(\lambda n.\ p\ n > 0)$ *sequentially*
**assumes** *divergent_p*: $\neg$*summable* $(\lambda n.\ inverse\ (p\ n))$
**defines** $l \equiv limsup\ (\lambda n.\ ereal\ (p\ n * f\ n\ /\ f\ (Suc\ n) - p\ (Suc\ n)))$
**assumes** *l*: $l < 0$
**shows** $\neg$*summable f*
**theorem** *ratio_test_convergence*:
**fixes** $f :: nat \Rightarrow real$
**assumes** *pos_f*: *eventually* $(\lambda n.\ f\ n > 0)$ *sequentially*
**defines** $l \equiv liminf\ (\lambda n.\ ereal\ (f\ n\ /\ f\ (Suc\ n)))$
**assumes** *l*: $l > 1$
**shows** *summable f*

**theorem** *ratio_test_divergence*:
**fixes** $f :: nat \Rightarrow real$
**assumes** *pos_f*: *eventually* $(\lambda n.\ f\ n > 0)$ *sequentially*
**defines** $l \equiv limsup\ (\lambda n.\ ereal\ (f\ n\ /\ f\ (Suc\ n)))$
**assumes** *l*: $l < 1$
**shows** $\neg$*summable f*
**theorem** *raabes_test_convergence*:
**fixes** $f :: nat \Rightarrow real$
**assumes** *pos*: *eventually* $(\lambda n.\ f\ n > 0)$ *sequentially*
**defines** $l \equiv liminf\ (\lambda n.\ ereal\ (of\_nat\ n * (f\ n\ /\ f\ (Suc\ n) - 1)))$
**assumes** *l*: $l > 1$
**shows** *summable f*

**theorem** *raabes_test_divergence*:
**fixes** $f :: nat \Rightarrow real$
**assumes** *pos*: *eventually* $(\lambda n.\ f\ n > 0)$ *sequentially*
**defines** $l \equiv limsup\ (\lambda n.\ ereal\ (of\_nat\ n * (f\ n\ /\ f\ (Suc\ n) - 1)))$
**assumes** *l*: $l < 1$
**shows** $\neg$*summable f*

### 5.14.2 Radius of convergence

**definition** *conv_radius* :: (*nat* ⇒ *'a* :: *banach*) ⇒ *ereal* **where**
*conv_radius f* = *inverse* (*limsup* (λ*n*. *ereal* (*root n* (*norm* (*f n*)))))

**theorem** *abs_summable_in_conv_radius*:
**fixes** *f* :: *nat* ⇒ *'a* :: {*banach*, *real_normed_div_algebra*}
**assumes** *ereal* (*norm z*) < *conv_radius f*
**shows** *summable* (λ*n*. *norm* (*f n* ∗ *z* ^ *n*))

**theorem** *not_summable_outside_conv_radius*:
**fixes** *f* :: *nat* ⇒ *'a* :: {*banach*, *real_normed_div_algebra*}
**assumes** *ereal* (*norm z*) > *conv_radius f*
**shows** ¬*summable* (λ*n*. *f n* ∗ *z* ^ *n*)

**end**

## 5.15 Uniform Limit and Uniform Convergence

**theory** *Uniform_Limit*
**imports** *Connected Summation_Tests Infinite_Sum*
**begin**

### 5.15.1 Definition

**definition** *uniformly_on* :: *'a set* ⇒ (*'a* ⇒ *'b*::*metric_space*) ⇒ (*'a* ⇒ *'b*) *filter*
**where** *uniformly_on S l* = (*INF e*∈{*0* <..}. *principal* {*f*. ∀ *x*∈*S*. *dist* (*f x*) (*l x*) < *e*})

**abbreviation**
*uniform_limit S f l* ≡ *filterlim f* (*uniformly_on S l*)

**proposition** *uniform_limit_iff*:
*uniform_limit S f l F* ⟷ (∀ *e*>*0*. ∀$_F$ *n in F*. ∀ *x*∈*S*. *dist* (*f n x*) (*l x*) < *e*)

### 5.15.2 Exchange limits

**proposition** *swap_uniform_limit*:
**assumes** *f*: ∀$_F$ *n in F*. (*f n* ⟶ *g n*) (*at x within S*)
**assumes** *g*: (*g* ⟶ *l*) *F*
**assumes** *uc*: *uniform_limit S f h F*
**assumes** ¬*trivial_limit F*
**shows** (*h* ⟶ *l*) (*at x within S*)

### 5.15.3 Uniform limit theorem

**theorem** *uniform_limit_theorem*:
  **assumes** *c*: $\forall_F$ *n in F. continuous_on A (f n)*
  **assumes** *ul*: *uniform_limit A f l F*
  **assumes** ¬ *trivial_limit F*
  **shows** *continuous_on A l*

### 5.15.4 Comparison Test

### 5.15.5 Weierstrass M-Test

**proposition** *Weierstrass_m_test_ev*:
**fixes** *f* :: _ ⇒ _ ⇒ _ :: *banach*
**assumes** *eventually* (λ*n*. ∀ *x*∈*A*. *norm* (*f n x*) ≤ *M n*) *sequentially*
**assumes** *summable M*
**shows** *uniform_limit A* (λ*n x*. ∑ *i*<*n*. *f i x*) (λ*x*. *suminf* (λ*i*. *f i x*)) *sequentially*

### 5.15.6 Power series and uniform convergence

**proposition** *powser_uniformly_convergent*:
  **fixes** *a* :: *nat* ⇒ ′*a*::{*real_normed_div_algebra*,*banach*}
  **assumes** *r* < *conv_radius a*
  **shows** *uniformly_convergent_on* (*cball* ξ *r*) (λ*n x*. ∑ *i*<*n*. *a i* ∗ (*x* − ξ) ^ *i*)

**end**

## 5.16 Bounded Linear Function

**theory** *Bounded_Linear_Function*
**imports**
  *Topology_Euclidean_Space*
  *Operator_Norm*
  *Uniform_Limit*
  *Function_Topology*

**begin**

### 5.16.1 Type of bounded linear functions

**typedef** (**overloaded**) (′*a*, ′*b*) *blinfun* ((_ $\Rightarrow_L$ /_) [*22*, *21*] *21*) =
  {*f*::′*a*::*real_normed_vector*⇒′*b*::*real_normed_vector*. *bounded_linear f*}
  **morphisms** *blinfun_apply Blinfun*

### 5.16.2 Type class instantiations

**instantiation** *blinfun* :: (*real_normed_vector*, *real_normed_vector*) *real_normed_vector*
**begin**

**lift_definition** *norm_blinfun* :: $'a \Rightarrow_L 'b \Rightarrow$ *real* **is** *onorm*
**lift_definition** *zero_blinfun* :: $'a \Rightarrow_L 'b$ **is** $\lambda x.\ 0$

**lift_definition** *plus_blinfun* :: $'a \Rightarrow_L 'b \Rightarrow 'a \Rightarrow_L 'b \Rightarrow 'a \Rightarrow_L 'b$
**is** $\lambda f\ g\ x.\ f\ x + g\ x$

**lift_definition** *scaleR_blinfun*::*real* $\Rightarrow 'a \Rightarrow_L 'b \Rightarrow 'a \Rightarrow_L 'b$ **is** $\lambda r\ f\ x.\ r *_R f\ x$

### 5.16.3 The strong operator topology on continuous linear operators

**definition** *strong_operator_topology*::($'a$::*real_normed_vector* $\Rightarrow_L 'b$::*real_normed_vector*) *topology*
**where** *strong_operator_topology* = *pullback_topology UNIV blinfun_apply euclidean*

**end**

## 5.17 Derivative

**theory** *Derivative*
**imports**
*Bounded_Linear_Function*
*Line_Segment*
*Convex_Euclidean_Space*
**begin**

### 5.17.1 Derivatives

**proposition** *has_derivative_within'*:
$(f$ *has_derivative* $f')($*at* $x$ *within* $s) \longleftrightarrow$
*bounded_linear* $f' \wedge$
$(\forall e>0.\ \exists d>0.\ \forall x'\in s.\ 0 < norm\ (x' - x) \wedge norm\ (x' - x) < d \longrightarrow$
$norm\ (f\ x' - f\ x - f'(x' - x))\ /\ norm\ (x' - x) < e)$

### 5.17.2 Differentiability

**definition**
*differentiable_on* :: ($'a$::*real_normed_vector* ⇒ $'b$::*real_normed_vector*) ⇒ $'a$ *set* ⇒ *bool*
(**infix** *differentiable′_on 50*)
**where** *f differentiable_on s* ⟷ (∀ *x*∈*s*. *f differentiable* (*at x within s*))

### 5.17.3 Frechet derivative and Jacobian matrix

**proposition** *frechet_derivative_works*:
*f differentiable net* ⟷ (*f has_derivative* (*frechet_derivative f net*)) *net*

### 5.17.4 Differentiability implies continuity

**proposition** *differentiable_imp_continuous_within*:
*f differentiable* (*at x within s*) ⟹ *continuous* (*at x within s*) *f*

### 5.17.5 The chain rule

**proposition** *diff_chain_within*[*derivative_intros*]:
**assumes** (*f has_derivative f′*) (*at x within s*)
**and** (*g has_derivative g′*) (*at* (*f x*) *within* (*f ‘ s*))
**shows** ((*g* ∘ *f*) *has_derivative* (*g′* ∘ *f′*))(*at x within s*)

### 5.17.6 Uniqueness of derivative

The general result is a bit messy because we need approachability of the limit point from any direction. But OK for nontrivial intervals etc.

**proposition** *frechet_derivative_unique_within*:
**fixes** *f* :: $'a$::*euclidean_space* ⇒ $'b$::*real_normed_vector*
**assumes** *1*: (*f has_derivative f′*) (*at x within S*)
**and** *2*: (*f has_derivative f″*) (*at x within S*)
**and** *S*: ⋀*i e*. ⟦*i*∈*Basis*; *e*>*0*⟧ ⟹ ∃ *d*. *0* < |*d*| ∧ |*d*| < *e* ∧ ($x + d *_R i$) ∈ *S*
**shows** *f′* = *f″*

**proposition** *frechet_derivative_unique_within_closed_interval*:
**fixes** *f*::$'a$::*euclidean_space* ⇒ $'b$::*real_normed_vector*
**assumes** *ab*: ⋀*i*. *i*∈*Basis* ⟹ *a*•*i* < *b*•*i*
**and** *x*: *x* ∈ *cbox a b*
**and** (*f has_derivative f′* ) (*at x within cbox a b*)
**and** (*f has_derivative f″*) (*at x within cbox a b*)
**shows** *f′* = *f″*

### 5.17.7 Derivatives of local minima and maxima are zero

### 5.17.8 One-dimensional mean value theorem

### 5.17.9 More general bound theorems

**proposition** *differentiable_bound_general*:
**fixes** $f$ :: *real* ⇒ *'a::real_normed_vector*
**assumes** $a < b$
**and** *f_cont*: *continuous_on* $\{a..b\}$ $f$
**and** *phi_cont*: *continuous_on* $\{a..b\}$ $\varphi$
**and** $f'$: $\bigwedge x.\ a < x \Longrightarrow x < b \Longrightarrow (f$ *has_vector_derivative* $f'\ x)$ (*at* $x$)
**and** *phi'*: $\bigwedge x.\ a < x \Longrightarrow x < b \Longrightarrow (\varphi$ *has_vector_derivative* $\varphi'\ x)$ (*at* $x$)
**and** *bnd*: $\bigwedge x.\ a < x \Longrightarrow x < b \Longrightarrow$ *norm* $(f'\ x) \le \varphi'\ x$
**shows** *norm* $(f\ b - f\ a) \le \varphi\ b - \varphi\ a$

### 5.17.10 Differentiability of inverse function (most basic form)

**proposition** *has_derivative_inverse*:
**fixes** $f$ :: *'a::real_normed_vector* ⇒ *'b::real_normed_vector*
**assumes** *compact* $S$
**and** $x \in S$
**and** *fx*: $f\ x \in$ *interior* ($f$ ‘ $S$)
**and** *continuous_on* $S\ f$
**and** *gf*: $\bigwedge y.\ y \in S \Longrightarrow g\ (f\ y) = y$
**and** *B*: ($f$ *has_derivative* $f'$) (*at* $x$) *bounded_linear* $g'$ $g' \circ f' = id$
**shows** ($g$ *has_derivative* $g'$) (*at* ($f\ x$))
**proposition** *has_derivative_locally_injective*:
**fixes** $f$ :: *'n::euclidean_space* ⇒ *'m::euclidean_space*
**assumes** $a \in S$
**and** *open* $S$
**and** *bling*: *bounded_linear* $g'$
**and** $g' \circ f'\ a = id$
**and** *derf*: $\bigwedge x.\ x \in S \Longrightarrow$ ($f$ *has_derivative* $f'\ x$) (*at* $x$)
**and** $\bigwedge e.\ e > 0 \Longrightarrow \exists d{>}0.\ \forall x.$ *dist* $a\ x < d \longrightarrow$ *onorm* $(\lambda v.\ f'\ x\ v - f'\ a\ v)$
$< e$
**obtains** $r$ **where** $r > 0$ *ball* $a\ r \subseteq S$ *inj_on* $f$ (*ball* $a\ r$)

### 5.17.11 Uniformly convergent sequence of derivatives

**proposition** *has_derivative_sequence*:
**fixes** $f$ :: *nat* ⇒ *'a::real_normed_vector* ⇒ *'b::banach*
**assumes** *convex* $S$
**and** *derf*: $\bigwedge n\ x.\ x \in S \Longrightarrow ((f\ n)$ *has_derivative* $(f'\ n\ x))$ (*at* $x$ *within* $S$)
**and** *nle*: $\bigwedge e.\ e > 0 \Longrightarrow \forall_F\ n$ *in sequentially*. $\forall x{\in}S.\ \forall h.$ *norm* $(f'\ n\ x\ h - g'$
$x\ h) \le e *$ *norm* $h$
**and** $x0 \in S$
**and** *lim*: $((\lambda n.\ f\ n\ x0) \longrightarrow l)$ *sequentially*

**shows** $\exists g. \forall x{\in}S. (\lambda n. f\ n\ x) \longrightarrow g\ x \wedge (g\ has\_derivative\ g'(x))\ (at\ x\ within\ S)$

### 5.17.12 Differentiation of a series

**proposition** *has_derivative_series*:
**fixes** $f :: nat \Rightarrow 'a::real\_normed\_vector \Rightarrow 'b::banach$
**assumes** *convex S*
**and** $\bigwedge n\ x.\ x \in S \Longrightarrow ((f\ n)\ has\_derivative\ (f'\ n\ x))\ (at\ x\ within\ S)$
**and** $\bigwedge e.\ e{>}0 \Longrightarrow \forall_F\ n\ in\ sequentially.\ \forall x{\in}S.\ \forall h.\ norm\ (sum\ (\lambda i.\ f'\ i\ x\ h)\ \{..{<}n\} - g'\ x\ h) \le e * norm\ h$
**and** $x \in S$
**and** $(\lambda n.\ f\ n\ x)\ sums\ l$
**shows** $\exists g. \forall x{\in}S. (\lambda n.\ f\ n\ x)\ sums\ (g\ x) \wedge (g\ has\_derivative\ g'\ x)\ (at\ x\ within\ S)$

### 5.17.13 Derivative as a vector

**proposition** *vector_derivative_works*:
$f\ differentiable\ net \longleftrightarrow (f\ has\_vector\_derivative\ (vector\_derivative\ f\ net))\ net$
(**is** $?l = ?r$)

### 5.17.14 Field differentiability

**definition** $field\_differentiable :: ['a \Rightarrow 'a::real\_normed\_field, 'a\ filter] \Rightarrow bool$
(**infixr** (*field'_differentiable*) *50*)
**where** $f\ field\_differentiable\ F \equiv \exists f'.\ (f\ has\_field\_derivative\ f')\ F$

### 5.17.15 Field derivative

**definition** $deriv :: ('a \Rightarrow 'a::real\_normed\_field) \Rightarrow 'a \Rightarrow 'a$ **where**
$deriv\ f\ x \equiv SOME\ D.\ DERIV\ f\ x :> D$

**proposition** *field_differentiable_derivI*:
$f\ field\_differentiable\ (at\ x) \Longrightarrow (f\ has\_field\_derivative\ deriv\ f\ x)\ (at\ x)$

### 5.17.16 Relation between convexity and derivative

**proposition** *convex_on_imp_above_tangent*:
**assumes** *convex*: *convex_on A f* **and** *connected*: *connected A*
**assumes** *c*: $c \in interior\ A$ **and** $x : x \in A$
**assumes** *deriv*: $(f\ has\_field\_derivative\ f')\ (at\ c\ within\ A)$
**shows** $f\ x - f\ c \ge f' * (x - c)$

### 5.17.17 Partial derivatives

**proposition** *has_derivative_partialsI*:
**fixes** $f$::'a::real_normed_vector ⇒ 'b::real_normed_vector ⇒ 'c::real_normed_vector
**assumes** *fx*: ((λx. f x y) has_derivative fx) (at x within X)
**assumes** *fy*: ⋀x y. x ∈ X ⟹ y ∈ Y ⟹ ((λy. f x y) has_derivative blinfun_apply (fy x y)) (at y within Y)
**assumes** *fy_cont*[unfolded continuous_within]: continuous (at (x, y) within X × Y) (λ(x, y). fy x y)
**assumes** y ∈ Y convex Y
**shows** ((λ(x, y). f x y) has_derivative (λ(tx, ty). fx tx + fy x y ty)) (at (x, y) within X × Y)

### 5.17.18 The Inverse Function Theorem

**theorem** *inverse_function_theorem*:
**fixes** f::'a::euclidean_space ⇒ 'a
**and** f'::'a ⇒ ('a ⇒$_L$ 'a)
**assumes** open U
**and** *derf*: ⋀x. x ∈ U ⟹ (f has_derivative (blinfun_apply (f' x))) (at x)
**and** *contf*: continuous_on U f'
**and** x0 ∈ U
**and** *invf*: invf o$_L$ f' x0 = id_blinfun
**obtains** U' V g g' **where** open U' U' ⊆ U x0 ∈ U' open V f x0 ∈ V homeomorphism U' V f g
⋀y. y ∈ V ⟹ (g has_derivative (g' y)) (at y)
⋀y. y ∈ V ⟹ g' y = inv (blinfun_apply (f'(g y)))
⋀y. y ∈ V ⟹ bij (blinfun_apply (f'(g y)))

### 5.17.19 The concept of continuously differentiable

**definition** *C1_differentiable_on* :: (real ⇒ 'a::real_normed_vector) ⇒ real set ⇒ bool
(**infix** *C1'_differentiable'_on 50*)
**where**
f C1_differentiable_on S ⟷
(∃ D. (∀ x ∈ S. (f has_vector_derivative (D x)) (at x)) ∧ continuous_on S D)

**definition** *piecewise_C1_differentiable_on*
(**infixr** *piecewise'_C1'_differentiable'_on 50*)
**where** f piecewise_C1_differentiable_on i ≡
continuous_on i f ∧
(∃ S. finite S ∧ (f C1_differentiable_on (i − S)))

**end**

## 5.18 Finite Cartesian Products of Euclidean Spaces

**theory** *Cartesian_Euclidean_Space*
**imports** *Derivative*
**begin**

### 5.18.1 Closures and interiors of halfspaces

### 5.18.2 Bounds on components etc. relative to operator norm

### 5.18.3 Convex Euclidean Space

### 5.18.4 Arbitrarily good rational approximations

**proposition** *matrix_rational_approximation*:
  **fixes** $A :: real^\frown{}'n^\frown{}'m$
  **assumes** $e > 0$
  **obtains** $B$ **where** $\bigwedge i\ j.\ B\$i\$j \in \mathbb{Q}$ $onorm(\lambda x.\ (A - B) *v\ x) < e$

### 5.18.5 Derivative

**definition** *jacobian f net* = *matrix*(*frechet_derivative f net*)

**proposition** *jacobian_works*:
  $(f::(real^\frown{}'a) \Rightarrow (real^\frown{}'b))$ *differentiable net* $\longleftrightarrow$
    (*f has_derivative* $(\lambda h.$ (*jacobian f net*) $*v\ h))$ *net* (**is** *?lhs* = *?rhs*)
**proposition** *differential_zero_maxmin_cart*:
  **fixes** $f::real^\frown{}'a \Rightarrow real^\frown{}'b$
  **assumes** $0 < e$ $((\forall y \in ball\ x\ e.\ (f\ y)\$k \leq (f\ x)\$k) \vee (\forall y{\in}ball\ x\ e.\ (f\ x)\$k \leq (f$
$y)\$k))$
    *f differentiable* (*at x*)
  **shows** *jacobian f* (*at x*) \$ $k = 0$

**end**

## 5.19 Bernstein-Weierstrass and Stone-Weierstrass

**theory** *Weierstrass_Theorems*
**imports** *Uniform_Limit Path_Connected Derivative*
**begin**

### 5.19.1 Bernstein polynomials

**definition** *Bernstein* :: [*nat*,*nat*,*real*] ⇒ *real* **where**
*Bernstein n k x* ≡ *of_nat* (*n choose k*) ∗ *x*^*k* ∗ (*1* − *x*)^(*n* − *k*)

### 5.19.2 Explicit Bernstein version of the 1D Weierstrass approximation theorem

**theorem** *Bernstein_Weierstrass*:
**fixes** *f* :: *real* ⇒ *real*
**assumes** *contf*: *continuous_on* {*0*..*1*} *f* **and** *e*: *0* < *e*
**shows** ∃ *N*. ∀ *n x*. *N* ≤ *n* ∧ *x* ∈ {*0*..*1*}
⟶ |*f x* − (∑ *k*≤*n*. *f*(*k*/*n*) ∗ *Bernstein n k x*)| < *e*

### 5.19.3 General Stone-Weierstrass theorem

**definition** *normf* :: ('*a*::*t2_space* ⇒ *real*) ⇒ *real*
**where** *normf f* ≡ *SUP x*∈*S*. |*f x*|
**proposition** (**in** *function_ring_on*) *Stone_Weierstrass_basic*:
**assumes** *f*: *continuous_on S f* **and** *e*: *e* > *0*
**shows** ∃ *g* ∈ *R*. ∀ *x*∈*S*. |*f x* − *g x*| < *e*

**theorem** (**in** *function_ring_on*) *Stone_Weierstrass*:
**assumes** *f*: *continuous_on S f*
**shows** ∃ *F*∈*UNIV* → *R*. *LIM n sequentially*. *F n* :> *uniformly_on S f*
**corollary** *Stone_Weierstrass_HOL*:
**fixes** *R* :: ('*a*::*t2_space* ⇒ *real*) *set* **and** *S* :: '*a set*
**assumes** *compact S* ⋀*c*. *P*(λ*x*. *c*::*real*)
⋀*f*. *P f* ⟹ *continuous_on S f*
⋀*f g*. *P*(*f*) ∧ *P*(*g*) ⟹ *P*(λ*x*. *f x* + *g x*) ⋀*f g*. *P*(*f*) ∧ *P*(*g*) ⟹ *P*(λ*x*. *f x* ∗ *g x*)
⋀*x y*. *x* ∈ *S* ∧ *y* ∈ *S* ∧ *x* ≠ *y* ⟹ ∃ *f*. *P*(*f*) ∧ *f x* ≠ *f y*
*continuous_on S f*
*0* < *e*
**shows** ∃ *g*. *P*(*g*) ∧ (∀ *x* ∈ *S*. |*f x* − *g x*| < *e*)

### 5.19.4 Polynomial functions

**definition** *polynomial_function* :: ('*a*::*real_normed_vector* ⇒ '*b*::*real_normed_vector*) ⇒ *bool*
**where**
*polynomial_function p* ≡ (∀ *f*. *bounded_linear f* ⟶ *real_polynomial_function* (*f o p*))

### 5.19.5 Stone-Weierstrass theorem for polynomial functions

**theorem** *Stone_Weierstrass_polynomial_function*:
  **fixes** *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
  **assumes** *S*: *compact S*
    **and** *f*: *continuous_on S f*
    **and** *e*: *0* < *e*
  **shows** ∃ *g*. *polynomial_function g* ∧ (∀ *x* ∈ *S*. *norm*(*f x* − *g x*) < *e*)

**proposition** *Stone_Weierstrass_uniform_limit*:
  **fixes** *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
  **assumes** *S*: *compact S*
  **and** *f*: *continuous_on S f*
  **obtains** *g* **where** *uniform_limit S g f sequentially* ⋀*n*. *polynomial_function* (*g n*)

### 5.19.6 Polynomial functions as paths

**proposition** *connected_open_polynomial_connected*:
  **fixes** *S* :: *'a*::*euclidean_space set*
  **assumes** *S*: *open S connected S*
    **and** *x* ∈ *S y* ∈ *S*
  **shows** ∃ *g*. *polynomial_function g* ∧ *path_image g* ⊆ *S* ∧ *pathstart g* = *x* ∧ *pathfinish g* = *y*

**theorem** *Stone_Weierstrass_polynomial_function_subspace*:
  **fixes** *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
  **assumes** *compact S*
    **and** *contf*: *continuous_on S f*
    **and** *0* < *e*
    **and** *subspace T f ‘ S* ⊆ *T*
  **obtains** *g* **where** *polynomial_function g g ‘ S* ⊆ *T*
    ⋀*x*. *x* ∈ *S* ⟹ *norm*(*f x* − *g x*) < *e*

**end**

# Chapter 6

# Measure and Integration Theory

**theory** *Sigma_Algebra*
**imports**
  *Complex_Main*
  *HOL−Library.Countable_Set*
  *HOL−Library.FuncSet*
  *HOL−Library.Indicator_Function*
  *HOL−Library.Extended_Nonnegative_Real*
  *HOL−Library.Disjoint_Sets*
**begin**

## 6.1 Sigma Algebra

### 6.1.1 Families of sets

**locale** *subset_class* =
  **fixes** $\Omega$ :: *'a set* **and** $M$ :: *'a set set*
  **assumes** *space_closed*: $M \subseteq Pow\ \Omega$
**locale** *semiring_of_sets* = *subset_class* +
  **assumes** *empty_sets*[*iff*]: $\{\} \in M$
  **assumes** *Int*[*intro*]: $\bigwedge a\ b.\ a \in M \Longrightarrow b \in M \Longrightarrow a \cap b \in M$
  **assumes** *Diff_cover*:
    $\bigwedge a\ b.\ a \in M \Longrightarrow b \in M \Longrightarrow \exists\, C{\subseteq}M.\ finite\ C \wedge disjoint\ C \wedge a - b = \bigcup C$
**locale** *ring_of_sets* = *semiring_of_sets* +
  **assumes** *Un* [*intro*]: $\bigwedge a\ b.\ a \in M \Longrightarrow b \in M \Longrightarrow a \cup b \in M$
**locale** *algebra* = *ring_of_sets* +
  **assumes** *top* [*iff*]: $\Omega \in M$

**proposition** *algebra_iff_Un*:
  *algebra* $\Omega$ $M \longleftrightarrow$
    $M \subseteq Pow\ \Omega\ \wedge$
    $\{\} \in M\ \wedge$
    $(\forall\, a \in M.\ \Omega - a \in M)\ \wedge$

(∀ a ∈ M. ∀ b ∈ M. a ∪ b ∈ M) (**is** _ ⟷ ?Un)

**proposition** *algebra_iff_Int*:
*algebra* Ω *M* ⟷
*M* ⊆ *Pow* Ω & {} ∈ *M* &
(∀ a ∈ M. Ω − a ∈ M) &
(∀ a ∈ M. ∀ b ∈ M. a ∩ b ∈ M) (**is** _ ⟷ ?Int)
**locale** *sigma_algebra* = *algebra* +
**assumes** *countable_nat_UN* [*intro*]: ⋀A. *range* A ⊆ M ⟹ (⋃i::nat. A i) ∈ M

Sigma algebras can naturally be created as the closure of any set of M with regard to the properties just postulated.

**inductive_set** *sigma_sets* :: 'a set ⇒ 'a set set ⇒ 'a set set
**for** *sp* :: 'a set **and** *A* :: 'a set set
**where**
*Basic*[*intro*, *simp*]: a ∈ A ⟹ a ∈ *sigma_sets sp A*
| *Empty*: {} ∈ *sigma_sets sp A*
| *Compl*: a ∈ *sigma_sets sp A* ⟹ sp − a ∈ *sigma_sets sp A*
| *Union*: (⋀i::nat. a i ∈ *sigma_sets sp A*) ⟹ (⋃ i. a i) ∈ *sigma_sets sp A*
**definition** *closed_cdi* :: 'a set ⇒ 'a set set ⇒ bool **where**
*closed_cdi* Ω M ⟷
M ⊆ *Pow* Ω &
(∀ s ∈ M. Ω − s ∈ M) &
(∀ A. (*range* A ⊆ M) & (A 0 = {}) & (∀ n. A n ⊆ A (*Suc* n)) ⟶
(⋃ i. A i) ∈ M) &
(∀ A. (*range* A ⊆ M) & *disjoint_family* A ⟶ (⋃ i::nat. A i) ∈ M)

**locale** *Dynkin_system* = *subset_class* +
**assumes** *space*: Ω ∈ M
**and** *compl*[*intro*!]: ⋀A. A ∈ M ⟹ Ω − A ∈ M
**and** *UN*[*intro*!]: ⋀A. *disjoint_family* A ⟹ *range* A ⊆ M
⟹ (⋃i::nat. A i) ∈ M
**definition** *Int_stable* :: 'a set set ⇒ bool **where**
*Int_stable* M ⟷ (∀ a ∈ M. ∀ b ∈ M. a ∩ b ∈ M)
**definition** *Dynkin* :: 'a set ⇒ 'a set set ⇒ 'a set set **where**
*Dynkin* Ω M = (⋂{D. *Dynkin_system* Ω D ∧ M ⊆ D})

The reason to introduce Dynkin-systems is the following induction rules for $\sigma$-algebras generated by a generator closed under intersection.

**proposition** *sigma_sets_induct_disjoint*[*consumes 3*, *case_names basic empty compl union*]:
**assumes** *Int_stable* G
**and** *closed*: G ⊆ *Pow* Ω
**and** *A*: A ∈ *sigma_sets* Ω G
**assumes** *basic*: ⋀A. A ∈ G ⟹ P A
**and** *empty*: P {}
**and** *compl*: ⋀A. A ∈ *sigma_sets* Ω G ⟹ P A ⟹ P (Ω − A)

**and** *union*: $\bigwedge A.$ *disjoint_family* $A \Longrightarrow$ *range* $A \subseteq$ *sigma_sets* $\Omega$ $G \Longrightarrow (\bigwedge i.$ $P$ $(A\ i)) \Longrightarrow P$ $(\bigcup i{::}nat.\ A\ i)$
**shows** $P\ A$

### 6.1.2 Measure type

**definition** *positive* :: $'a$ *set set* $\Rightarrow$ $('a$ *set* $\Rightarrow$ *ennreal*$) \Rightarrow$ *bool* **where**
*positive* $M\ \mu \longleftrightarrow \mu\ \{\} = 0$

**definition** *countably_additive* :: $'a$ *set set* $\Rightarrow$ $('a$ *set* $\Rightarrow$ *ennreal*$) \Rightarrow$ *bool* **where**
*countably_additive* $M\ f \longleftrightarrow$
$(\forall A.$ *range* $A \subseteq M \longrightarrow$ *disjoint_family* $A \longrightarrow (\bigcup i.\ A\ i) \in M \longrightarrow$
$(\sum i.\ f\ (A\ i)) = f\ (\bigcup i.\ A\ i))$

**definition** *measure_space* :: $'a$ *set* $\Rightarrow$ $'a$ *set set* $\Rightarrow$ $('a$ *set* $\Rightarrow$ *ennreal*$) \Rightarrow$ *bool* **where**
*measure_space* $\Omega\ A\ \mu \longleftrightarrow$
*sigma_algebra* $\Omega\ A \wedge$ *positive* $A\ \mu \wedge$ *countably_additive* $A\ \mu$

**typedef** $'a$ *measure* =
$\{(\Omega{::}'a\ set,\ A,\ \mu).\ (\forall a{\in}{-}A.\ \mu\ a = 0) \wedge$ *measure_space* $\Omega\ A\ \mu\ \}$

**definition** *space* :: $'a$ *measure* $\Rightarrow$ $'a$ *set* **where**
*space* $M$ = *fst* (*Rep_measure* $M$)

**definition** *sets* :: $'a$ *measure* $\Rightarrow$ $'a$ *set set* **where**
*sets* $M$ = *fst* (*snd* (*Rep_measure* $M$))

**definition** *emeasure* :: $'a$ *measure* $\Rightarrow$ $'a$ *set* $\Rightarrow$ *ennreal* **where**
*emeasure* $M$ = *snd* (*snd* (*Rep_measure* $M$))

**definition** *measure* :: $'a$ *measure* $\Rightarrow$ $'a$ *set* $\Rightarrow$ *real* **where**
*measure* $M\ A$ = *enn2real* (*emeasure* $M\ A$)

**definition** *measure_of* :: $'a$ *set* $\Rightarrow$ $'a$ *set set* $\Rightarrow$ $('a$ *set* $\Rightarrow$ *ennreal*$) \Rightarrow$ $'a$ *measure*
**where**
*measure_of* $\Omega\ A\ \mu$ =
*Abs_measure* $(\Omega,$ *if* $A \subseteq$ *Pow* $\Omega$ *then* *sigma_sets* $\Omega\ A$ *else* $\{\{\}, \Omega\},$
$\lambda a.$ *if* $a \in$ *sigma_sets* $\Omega\ A \wedge$ *measure_space* $\Omega$ (*sigma_sets* $\Omega\ A$) $\mu$ *then* $\mu\ a$ *else* $0)$
**proposition** *emeasure_measure_of*:
**assumes** $M$: $M$ = *measure_of* $\Omega\ A\ \mu$
**assumes** *ms*: $A \subseteq$ *Pow* $\Omega$ *positive* (*sets* $M$) $\mu$ *countably_additive* (*sets* $M$) $\mu$
**assumes** $X$: $X \in$ *sets* $M$
**shows** *emeasure* $M\ X = \mu\ X$
**definition** *measurable* :: $'a$ *measure* $\Rightarrow$ $'b$ *measure* $\Rightarrow$ $('a \Rightarrow 'b)$ *set*
(**infixr** $\rightarrow_M$ *60*) **where**
*measurable* $A\ B = \{f \in$ *space* $A \rightarrow$ *space* $B.\ \forall y \in$ *sets* $B.\ f$ -' $y \cap$ *space* $A \in$ *sets*

*A*}
**definition** *count_space* :: *'a set* ⇒ *'a measure* **where**
*count_space* Ω = *measure_of* Ω (*Pow* Ω) (λ*A*. *if finite A then of_nat* (*card A*) *else* ∞)

### 6.1.3 The smallest σ-algebra regarding a function

**definition** *vimage_algebra* :: *'a set* ⇒ (*'a* ⇒ *'b*) ⇒ *'b measure* ⇒ *'a measure* **where**
*vimage_algebra X f M* = *sigma X* {*f* −' *A* ∩ *X* | *A*. *A* ∈ *sets M*}

**end**

## 6.2 Measurability Prover

**theory** *Measurable*
**imports**
*Sigma_Algebra*
*HOL−Library.Order_Continuity*
**begin**

**method_setup** *measurable* = ‹ *Scan.lift* (*Scan.succeed* (*METHOD o Measurable.measurable_tac*)) ›
*measurability prover*

**simproc_setup** *measurable* (*A* ∈ *sets M* | *f* ∈ *measurable M N*) = ‹*K Measurable.simproc*›

**end**

## 6.3 Measure Spaces

**theory** *Measure_Space*
**imports**
*Measurable HOL−Library.Extended_Nonnegative_Real*
**begin**

### 6.3.1 μ-null sets

**definition** *null_sets* :: *'a measure* ⇒ *'a set set* **where**
*null_sets M* = {*N*∈*sets M*. *emeasure M N* = *0*}

### 6.3.2 The almost everywhere filter (i.e. quantifier)

**definition** *ae_filter* :: *'a measure* ⇒ *'a filter* **where**
*ae_filter M* = (*INF N*∈*null_sets M*. *principal* (*space M* − *N*))

### 6.3.3 $\sigma$-finite Measures

**locale** *sigma_finite_measure* =
**fixes** $M$ :: $'a$ *measure*
**assumes** *sigma_finite_countable*:
$\exists A::'a\ set\ set.\ countable\ A \wedge A \subseteq sets\ M \wedge (\bigcup A) = space\ M \wedge (\forall a \in A.\ emeasure\ M\ a \neq \infty)$

### 6.3.4 Measure space induced by distribution of $(\rightarrow_M)$-functions

**definition** *distr* :: $'a\ measure \Rightarrow\ 'b\ measure \Rightarrow ('a \Rightarrow\ 'b) \Rightarrow\ 'b\ measure$ **where**
$distr\ M\ N\ f =$
$measure\_of\ (space\ N)\ (sets\ N)\ (\lambda A.\ emeasure\ M\ (f -`\ A \cap space\ M))$

**proposition** *distr_distr*:
$g \in measurable\ N\ L \Longrightarrow f \in measurable\ M\ N \Longrightarrow distr\ (distr\ M\ N\ f)\ L\ g = distr\ M\ L\ (g \circ f)$

### 6.3.5 Set of measurable sets with finite measure

**definition** *fmeasurable* :: $'a\ measure \Rightarrow\ 'a\ set\ set$ **where**
$fmeasurable\ M = \{A \in sets\ M.\ emeasure\ M\ A < \infty\}$

### 6.3.6 Measure spaces with $emeasure\ M\ (space\ M) < \infty$

**locale** *finite_measure* = *sigma_finite_measure* $M$ **for** $M$ +
**assumes** *finite_emeasure_space*: $emeasure\ M\ (space\ M) \neq top$

### 6.3.7 Scaling a measure

**definition** *scale_measure* :: $ennreal \Rightarrow\ 'a\ measure \Rightarrow\ 'a\ measure$ **where**
$scale\_measure\ r\ M = measure\_of\ (space\ M)\ (sets\ M)\ (\lambda A.\ r * emeasure\ M\ A)$

### 6.3.8 Complete lattice structure on measures

**proposition** *unsigned_Hahn_decomposition*:
**assumes** [*simp*]: $sets\ N = sets\ M$ **and** [*measurable*]: $A \in sets\ M$
**and** [*simp*]: $emeasure\ M\ A \neq top\ emeasure\ N\ A \neq top$
**shows** $\exists Y \in sets\ M.\ Y \subseteq A \wedge (\forall X \in sets\ M.\ X \subseteq Y \longrightarrow N\ X \leq M\ X) \wedge (\forall X \in sets\ M.\ X \subseteq A \longrightarrow X \cap Y = \{\} \longrightarrow M\ X \leq N\ X)$

Define a lexicographical order on *measure*, in the order space, sets and measure. The parts of the lexicographical order are point-wise ordered.

**instantiation** *measure* :: (*type*) *order_bot*
**begin**

**definition** *less_measure* :: *'a measure* ⇒ *'a measure* ⇒ *bool* **where**
*less_measure M N* ⟷ (*M* ≤ *N* ∧ ¬ *N* ≤ *M*)

**definition** *bot_measure* :: *'a measure* **where**
*bot_measure* = *sigma* {} {}

**proposition** *le_measure*: *sets M* = *sets N* ⟹ *M* ≤ *N* ⟷ (∀ *A*∈*sets M*. *emeasure M A* ≤ *emeasure N A*)

**definition** *sup_measure'* :: *'a measure* ⇒ *'a measure* ⇒ *'a measure* **where**
*sup_measure' A B* =
*measure_of* (*space A*) (*sets A*)
(λ*X*. *SUP Y*∈*sets A*. *emeasure A* (*X* ∩ *Y*) + *emeasure B* (*X* ∩ − *Y*))

**definition** *sup_lexord* :: *'a* ⇒ *'a* ⇒ (*'a* ⇒ *'b*::*order*) ⇒ *'a* ⇒ *'a* ⇒ *'a* **where**
*sup_lexord A B k s c* =
(*if k A* = *k B then c else*
*if* ¬ *k A* ≤ *k B* ∧ ¬ *k B* ≤ *k A then s else*
*if k B* ≤ *k A then A else B*)

**instantiation** *measure* :: (*type*) *semilattice_sup*
**begin**

**definition** *sup_measure* :: *'a measure* ⇒ *'a measure* ⇒ *'a measure* **where**
*sup_measure A B* =
*sup_lexord A B space* (*sigma* (*space A* ∪ *space B*) {})
(*sup_lexord A B sets* (*sigma* (*space A*) (*sets A* ∪ *sets B*)) (*sup_measure' A B*))

**definition**
*Sup_lexord* :: (*'a* ⇒ *'b*::*complete_lattice*) ⇒ (*'a set* ⇒ *'a*) ⇒ (*'a set* ⇒ *'a*) ⇒ *'a set* ⇒ *'a*
**where**
*Sup_lexord k c s A* =
(*let U* = (*SUP a*∈*A*. *k a*)
*in if* ∃ *a*∈*A*. *k a* = *U then c* {*a*∈*A*. *k a* = *U*} *else s A*)

**instantiation** *measure* :: (*type*) *complete_lattice*
**begin**

**definition** *Sup_measure'* :: *'a measure set* ⇒ *'a measure* **where**
*Sup_measure' M* =
*measure_of* (⋃ *a*∈*M*. *space a*) (⋃ *a*∈*M*. *sets a*)
(λ*X*. (*SUP P*∈{*P*. *finite P* ∧ *P* ⊆ *M* }. *sup_measure.F id P X*))

**definition** *Sup_measure* :: *'a measure set* ⇒ *'a measure* **where**
*Sup_measure* =
  *Sup_lexord space*
    (*Sup_lexord sets Sup_measure'*
      ($\lambda U$. *sigma* ($\bigcup u \in U$. *space u*) ($\bigcup u \in U$. *sets u*)))
    ($\lambda U$. *sigma* ($\bigcup u \in U$. *space u*) {})

**definition** *Inf_measure* :: *'a measure set* ⇒ *'a measure* **where**
  *Inf_measure A* = *Sup* {*x*. ∀ *a*∈*A*. *x* ≤ *a*}

**definition** *inf_measure* :: *'a measure* ⇒ *'a measure* ⇒ *'a measure* **where**
  *inf_measure a b* = *Inf* {*a*, *b*}

**definition** *top_measure* :: *'a measure* **where**
  *top_measure* = *Inf* {}

**end**

## 6.4 Borel Space

**theory** *Borel_Space*
**imports**
  *Measurable Derivative Ordered_Euclidean_Space Extended_Real_Limits*
**begin**

**proposition** *open_prod_generated*: *open* = *generate_topology* {*A* × *B* | *A B*. *open A* ∧ *open B*}

**proposition** *mono_on_imp_deriv_nonneg*:
  **assumes** *mono*: *mono_on A f* **and** *deriv*: (*f has_real_derivative D*) (*at x*)
  **assumes** *x* ∈ *interior A*
  **shows** *D* ≥ *0*

**proposition** *mono_on_ctble_discont*:
  **fixes** *f* :: *real* ⇒ *real*
  **fixes** *A* :: *real set*
  **assumes** *mono_on A f*
  **shows** *countable* {*a*∈*A*. ¬ *continuous* (*at a within A*) *f*}

### 6.4.1 Generic Borel spaces

**definition** (**in** *topological_space*) *borel* :: *'a measure* **where**
  *borel* = *sigma UNIV* {*S*. *open S*}

**theorem** *second_countable_borel_measurable*:
**fixes** $X :: 'a::second\_countable\_topology\ set\ set$
**assumes** *eq*: $open = generate\_topology\ X$
**shows** $borel = sigma\ UNIV\ X$

**proposition** *borel_eq_countable_basis*:
**fixes** $B::'a::topological\_space\ set\ set$
**assumes** *countable B*
**assumes** *topological_basis B*
**shows** $borel = sigma\ UNIV\ B$

### 6.4.2 Borel spaces on order topologies

### 6.4.3 Borel spaces on topological monoids

### 6.4.4 Borel spaces on Euclidean spaces

### 6.4.5 Borel measurable operators

**lemma** *borel_measurable_complex_iff*:
$f \in borel\_measurable\ M \longleftrightarrow$
$(\lambda x.\ Re\ (f\ x)) \in borel\_measurable\ M \wedge (\lambda x.\ Im\ (f\ x)) \in borel\_measurable\ M$

### 6.4.6 Borel space on the extended reals

**theorem** *borel_measurable_ereal_iff_real*:
**fixes** $f :: 'a \Rightarrow ereal$
**shows** $f \in borel\_measurable\ M \longleftrightarrow$
$((\lambda x.\ real\_of\_ereal\ (f\ x)) \in borel\_measurable\ M \wedge f -` \{\infty\} \cap space\ M \in sets\ M \wedge f -` \{-\infty\} \cap space\ M \in sets\ M)$

### 6.4.7 Borel space on the extended non-negative reals

**definition** [*simp*]: $is\_borel\ f\ M \longleftrightarrow f \in borel\_measurable\ M$

### 6.4.8 LIMSEQ is borel measurable

**proposition** *measurable_limit* [*measurable*]:
**fixes** $f::nat \Rightarrow 'a \Rightarrow 'b::first\_countable\_topology$
**assumes** [*measurable*]: $\bigwedge n::nat.\ f\ n \in borel\_measurable\ M$
**shows** $Measurable.pred\ M\ (\lambda x.\ (\lambda n.\ f\ n\ x) \longlongrightarrow c)$

**end**

## 6.5 Lebesgue Integration for Nonnegative Functions

**theory** *Nonnegative_Lebesgue_Integration*
**imports** *Measure_Space Borel_Space*
**begin**

### 6.5.1 Simple function

**definition** $simple\_function\ M\ g \longleftrightarrow$
$finite\ (g\ `\ space\ M) \wedge$
$(\forall x \in g\ `\ space\ M.\ g\ -`\ \{x\} \cap space\ M \in sets\ M)$

**lemma** $borel\_measurable\_implies\_simple\_function\_sequence$:
**fixes** $u :: 'a \Rightarrow ennreal$
**assumes** $u[measurable]$: $u \in borel\_measurable\ M$
**shows** $\exists f.\ incseq\ f \wedge (\forall i.\ (\forall x.\ f\ i\ x < top) \wedge simple\_function\ M\ (f\ i)) \wedge u = (SUP\ i.\ f\ i)$

**lemma** $simple\_function\_induct$
$[consumes\ 1,\ case\_names\ cong\ set\ mult\ add,\ induct\ set:\ simple\_function]$:
**fixes** $u :: 'a \Rightarrow ennreal$
**assumes** $u$: $simple\_function\ M\ u$
**assumes** $cong$: $\bigwedge f\ g.\ simple\_function\ M\ f \Longrightarrow simple\_function\ M\ g \Longrightarrow (AE\ x\ in\ M.\ f\ x = g\ x) \Longrightarrow P\ f \Longrightarrow P\ g$
**assumes** $set$: $\bigwedge A.\ A \in sets\ M \Longrightarrow P\ (indicator\ A)$
**assumes** $mult$: $\bigwedge u\ c.\ P\ u \Longrightarrow P\ (\lambda x.\ c * u\ x)$
**assumes** $add$: $\bigwedge u\ v.\ P\ u \Longrightarrow P\ v \Longrightarrow P\ (\lambda x.\ v\ x + u\ x)$
**shows** $P\ u$

**lemma** $borel\_measurable\_induct$
$[consumes\ 1,\ case\_names\ cong\ set\ mult\ add\ seq,\ induct\ set:\ borel\_measurable]$:
**fixes** $u :: 'a \Rightarrow ennreal$
**assumes** $u$: $u \in borel\_measurable\ M$
**assumes** $cong$: $\bigwedge f\ g.\ f \in borel\_measurable\ M \Longrightarrow g \in borel\_measurable\ M \Longrightarrow (\bigwedge x.\ x \in space\ M \Longrightarrow f\ x = g\ x) \Longrightarrow P\ g \Longrightarrow P\ f$
**assumes** $set$: $\bigwedge A.\ A \in sets\ M \Longrightarrow P\ (indicator\ A)$
**assumes** $mult'$: $\bigwedge u\ c.\ c < top \Longrightarrow u \in borel\_measurable\ M \Longrightarrow (\bigwedge x.\ x \in space\ M \Longrightarrow u\ x < top) \Longrightarrow P\ u \Longrightarrow P\ (\lambda x.\ c * u\ x)$
**assumes** $add$: $\bigwedge u\ v.\ u \in borel\_measurable\ M \Longrightarrow (\bigwedge x.\ x \in space\ M \Longrightarrow u\ x < top) \Longrightarrow P\ u \Longrightarrow v \in borel\_measurable\ M \Longrightarrow (\bigwedge x.\ x \in space\ M \Longrightarrow v\ x < top) \Longrightarrow (\bigwedge x.\ x \in space\ M \Longrightarrow u\ x = 0 \vee v\ x = 0) \Longrightarrow P\ v \Longrightarrow P\ (\lambda x.\ v\ x + u\ x)$
**assumes** $seq$: $\bigwedge U.\ (\bigwedge i.\ U\ i \in borel\_measurable\ M) \Longrightarrow (\bigwedge i\ x.\ x \in space\ M \Longrightarrow U\ i\ x < top) \Longrightarrow (\bigwedge i.\ P\ (U\ i)) \Longrightarrow incseq\ U \Longrightarrow u = (SUP\ i.\ U\ i) \Longrightarrow P\ (SUP\ i.\ U\ i)$

**shows** $P\ u$

### 6.5.2 Simple integral

**definition** $simple\_integral$ :: $'a\ measure \Rightarrow ('a \Rightarrow ennreal) \Rightarrow ennreal\ (integral^S)$
**where**
$integral^S\ M\ f = (\sum x \in f\ `\ space\ M.\ x * emeasure\ M\ (f\ -`\ \{x\} \cap space\ M))$

### 6.5.3 Integral on nonnegative functions

**definition** $nn\_integral$ :: $'a\ measure \Rightarrow ('a \Rightarrow ennreal) \Rightarrow ennreal\ (integral^N)$
**where**
$integral^N\ M\ f = (SUP\ g \in \{g.\ simple\_function\ M\ g \wedge g \leq f\}.\ integral^S\ M\ g)$

**theorem** $nn\_integral\_monotone\_convergence\_SUP\_AE$:
**assumes** $f$: $\bigwedge i.\ AE\ x\ in\ M.\ f\ i\ x \leq f\ (Suc\ i)\ x\ \bigwedge i.\ f\ i \in borel\_measurable\ M$
**shows** $(\int^+ x.\ (SUP\ i.\ f\ i\ x)\ \partial M) = (SUP\ i.\ integral^N\ M\ (f\ i))$

**theorem** $nn\_integral\_suminf$:
**assumes** $f$: $\bigwedge i.\ f\ i \in borel\_measurable\ M$
**shows** $(\int^+ x.\ (\sum i.\ f\ i\ x)\ \partial M) = (\sum i.\ integral^N\ M\ (f\ i))$

**theorem** $nn\_integral\_Markov\_inequality$:
**assumes** $u$: $(\lambda x.\ u\ x * indicator\ A\ x) \in borel\_measurable\ M$ **and** $A \in sets\ M$
**shows** $(emeasure\ M)\ (\{x{\in}A.\ 1 \leq c * u\ x\}) \leq c * (\int^+ x.\ u\ x * indicator\ A\ x$
$\partial M)$
(**is** $(emeasure\ M)\ ?A \leq \_ * ?PI$)

**theorem** $nn\_integral\_monotone\_convergence\_INF\_AE$:
**fixes** $f$ :: $nat \Rightarrow 'a \Rightarrow ennreal$
**assumes** $f$: $\bigwedge i.\ AE\ x\ in\ M.\ f\ (Suc\ i)\ x \leq f\ i\ x$
**and** $[measurable]$: $\bigwedge i.\ f\ i \in borel\_measurable\ M$
**and** $fin$: $(\int^+ x.\ f\ i\ x\ \partial M) < \infty$
**shows** $(\int^+ x.\ (INF\ i.\ f\ i\ x)\ \partial M) = (INF\ i.\ integral^N\ M\ (f\ i))$

**theorem** $nn\_integral\_liminf$:
**fixes** $u$ :: $nat \Rightarrow 'a \Rightarrow ennreal$
**assumes** $u$: $\bigwedge i.\ u\ i \in borel\_measurable\ M$
**shows** $(\int^+ x.\ liminf\ (\lambda n.\ u\ n\ x)\ \partial M) \leq liminf\ (\lambda n.\ integral^N\ M\ (u\ n))$

**theorem** $nn\_integral\_limsup$:
**fixes** $u$ :: $nat \Rightarrow 'a \Rightarrow ennreal$
**assumes** $[measurable]$: $\bigwedge i.\ u\ i \in borel\_measurable\ M\ w \in borel\_measurable\ M$
**assumes** $bounds$: $\bigwedge i.\ AE\ x\ in\ M.\ u\ i\ x \leq w\ x$ **and** $w$: $(\int^+ x.\ w\ x\ \partial M) < \infty$
**shows** $limsup\ (\lambda n.\ integral^N\ M\ (u\ n)) \leq (\int^+ x.\ limsup\ (\lambda n.\ u\ n\ x)\ \partial M)$

**theorem** $nn\_integral\_dominated\_convergence$:
**assumes** $[measurable]$:

$\bigwedge i.$ *u i* ∈ *borel_measurable M u′* ∈ *borel_measurable M w* ∈ *borel_measurable M*
**and** *bound*: $\bigwedge j.$ *AE x in M. u j x* ≤ *w x*
**and** *w*: (∫ ⁺*x. w x* ∂*M*) < ∞
**and** *u′*: *AE x in M.* (λ*i. u i x*) ⟶ *u′ x*
**shows** (λ*i.* (∫ ⁺*x. u i x* ∂*M*)) ⟶ (∫ ⁺*x. u′ x* ∂*M*)

**theorem** *nn_integral_lfp*:
**assumes** *sets*[*simp*]: $\bigwedge s.$ *sets* (*M s*) = *sets N*
**assumes** *f*: *sup_continuous f*
**assumes** *g*: *sup_continuous g*
**assumes** *meas*: $\bigwedge F.$ *F* ∈ *borel_measurable N* ⟹ *f F* ∈ *borel_measurable N*
**assumes** *step*: $\bigwedge F\ s.$ *F* ∈ *borel_measurable N* ⟹ *integral*$^N$ (*M s*) (*f F*) = *g* (λ*s. integral*$^N$ (*M s*) *F*) *s*
**shows** (∫ ⁺ω. *lfp f* ω ∂*M s*) = *lfp g s*

**theorem** *nn_integral_gfp*:
**assumes** *sets*[*simp*]: $\bigwedge s.$ *sets* (*M s*) = *sets N*
**assumes** *f*: *inf_continuous f* **and** *g*: *inf_continuous g*
**assumes** *meas*: $\bigwedge F.$ *F* ∈ *borel_measurable N* ⟹ *f F* ∈ *borel_measurable N*
**assumes** *bound*: $\bigwedge F\ s.$ *F* ∈ *borel_measurable N* ⟹ (∫ ⁺*x. f F x* ∂*M s*) < ∞
**assumes** *non_zero*: $\bigwedge s.$ *emeasure* (*M s*) (*space* (*M s*)) ≠ *0*
**assumes** *step*: $\bigwedge F\ s.$ *F* ∈ *borel_measurable N* ⟹ *integral*$^N$ (*M s*) (*f F*) = *g* (λ*s. integral*$^N$ (*M s*) *F*) *s*
**shows** (∫ ⁺ω. *gfp f* ω ∂*M s*) = *gfp g s*

### 6.5.4 Integral under concrete measures

**definition** *density* :: ′*a measure* ⇒ (′*a* ⇒ *ennreal*) ⇒ ′*a measure* **where**
*density M f* = *measure_of* (*space M*) (*sets M*) (λ*A.* ∫ ⁺ *x. f x* ∗ *indicator A x* ∂*M*)

**lemma** *nn_integral_density*:
**assumes** *f*: *f* ∈ *borel_measurable M*
**assumes** *g*: *g* ∈ *borel_measurable M*
**shows** *integral*$^N$ (*density M f*) *g* = (∫ ⁺ *x. f x* ∗ *g x* ∂*M*)
**definition** *point_measure* :: ′*a set* ⇒ (′*a* ⇒ *ennreal*) ⇒ ′*a measure* **where**
*point_measure A f* = *density* (*count_space A*) *f*
**definition** *uniform_measure M A* = *density M* (λ*x. indicator A x* / *emeasure M A*)
**definition** *uniform_count_measure A* = *point_measure A* (λ*x. 1* / *card A*)

**end**

## 6.6 Binary Product Measure

**theory** *Binary_Product_Measure*

**imports** *Nonnegative_Lebesgue_Integration*
**begin**

### 6.6.1 Binary products

**definition** *pair_measure* (**infixr** $\bigotimes_M$ *80*) **where**
$A \bigotimes_M B = measure\_of\ (space\ A \times space\ B)$
$\{a \times b \mid a\ b.\ a \in sets\ A \wedge b \in sets\ B\}$
$(\lambda X.\ \int^+ x.\ (\int^+ y.\ indicator\ X\ (x,y)\ \partial B)\ \partial A)$

**proposition** (**in** *sigma_finite_measure*) *emeasure_pair_measure_Times*:
**assumes** $A$: $A \in sets\ N$ **and** $B$: $B \in sets\ M$
**shows** $emeasure\ (N \bigotimes_M M)\ (A \times B) = emeasure\ N\ A * emeasure\ M\ B$

### 6.6.2 Binary products of σ-finite emeasure spaces

**proposition** (**in** *pair_sigma_finite*) *sigma_finite_up_in_pair_measure_generator*:
**defines** $E \equiv \{A \times B \mid A\ B.\ A \in sets\ M1 \wedge B \in sets\ M2\}$
**shows** $\exists F{::}nat \Rightarrow ('a \times 'b)\ set.\ range\ F \subseteq E \wedge incseq\ F \wedge (\bigcup i.\ F\ i) = space\ M1 \times space\ M2\ \wedge$
$(\forall i.\ emeasure\ (M1 \bigotimes_M M2)\ (F\ i) \neq \infty)$

### 6.6.3 Fubinis theorem

**proposition** (**in** *pair_sigma_finite*) *nn_integral_snd*:
**assumes** $f[measurable]$: $f \in borel\_measurable\ (M1 \bigotimes_M M2)$
**shows** $(\int^+ y.\ (\int^+ x.\ f\ (x,\ y)\ \partial M1)\ \partial M2) = integral^N\ (M1 \bigotimes_M M2)\ f$

**theorem** (**in** *pair_sigma_finite*) *Fubini*:
**assumes** $f$: $f \in borel\_measurable\ (M1 \bigotimes_M M2)$
**shows** $(\int^+ y.\ (\int^+ x.\ f\ (x,\ y)\ \partial M1)\ \partial M2) = (\int^+ x.\ (\int^+ y.\ f\ (x,\ y)\ \partial M2)\ \partial M1)$

**theorem** (**in** *pair_sigma_finite*) *Fubini'*:
**assumes** $f$: $case\_prod\ f \in borel\_measurable\ (M1 \bigotimes_M M2)$
**shows** $(\int^+ y.\ (\int^+ x.\ f\ x\ y\ \partial M1)\ \partial M2) = (\int^+ x.\ (\int^+ y.\ f\ x\ y\ \partial M2)\ \partial M1)$

### 6.6.4 Products on counting spaces, densities and distributions

**proposition** *sigma_prod*:
**assumes** *X_cover*: $\exists E \subseteq A.\ countable\ E \wedge X = \bigcup E$ **and** $A$: $A \subseteq Pow\ X$
**assumes** *Y_cover*: $\exists E \subseteq B.\ countable\ E \wedge Y = \bigcup E$ **and** $B$: $B \subseteq Pow\ Y$

**shows** *sigma X A* $\bigotimes_M$ *sigma Y B* = *sigma* (*X* × *Y*) {*a* × *b* | *a b*. *a* ∈ *A* ∧ *b* ∈ *B*}
(**is** *?P* = *?S*)

**proposition** *sets_pair_eq*:
**assumes** *Ea*: *Ea* ⊆ *Pow* (*space A*) *sets A* = *sigma_sets* (*space A*) *Ea*
**and** *Ca*: *countable Ca Ca* ⊆ *Ea* ⋃ *Ca* = *space A*
**and** *Eb*: *Eb* ⊆ *Pow* (*space B*) *sets B* = *sigma_sets* (*space B*) *Eb*
**and** *Cb*: *countable Cb Cb* ⊆ *Eb* ⋃ *Cb* = *space B*
**shows** *sets* (*A* $\bigotimes_M$ *B*) = *sets* (*sigma* (*space A* × *space B*) { *a* × *b* | *a b*. *a* ∈ *Ea* ∧ *b* ∈ *Eb* })
(**is** _ = *sets* (*sigma* *?Ω* *?E*))

**proposition** *borel_prod*:
(*borel* $\bigotimes_M$ *borel*) = (*borel* :: (*'a*::*second_countable_topology* × *'b*::*second_countable_topology*) *measure*)
(**is** *?P* = *?B*)

**proposition** *pair_measure_count_space*:
**assumes** *A*: *finite A* **and** *B*: *finite B*
**shows** *count_space A* $\bigotimes_M$ *count_space B* = *count_space* (*A* × *B*) (**is** *?P* = *?C*)

**theorem** *pair_measure_density*:
**assumes** *f*: *f* ∈ *borel_measurable M1*
**assumes** *g*: *g* ∈ *borel_measurable M2*
**assumes** *sigma_finite_measure M2 sigma_finite_measure* (*density M2 g*)
**shows** *density M1 f* $\bigotimes_M$ *density M2 g* = *density* (*M1* $\bigotimes_M$ *M2*) (λ(*x*,*y*). *f x* ∗ *g y*) (**is** *?L* = *?R*)

**proposition** *nn_integral_fst_count_space*:
($\int^+$ *x*. $\int^+$ *y*. *f* (*x*, *y*) ∂*count_space UNIV* ∂*count_space UNIV*) = $integral^N$ (*count_space UNIV*) *f*
(**is** *?lhs* = *?rhs*)

**proposition** *nn_integral_snd_count_space*:
($\int^+$ *y*. $\int^+$ *x*. *f* (*x*, *y*) ∂*count_space UNIV* ∂*count_space UNIV*) = $integral^N$ (*count_space UNIV*) *f*
(**is** *?lhs* = *?rhs*)

### 6.6.5 Product of Borel spaces

**theorem** *borel_Times*:
**fixes** *A* :: *'a*::*topological_space set* **and** *B* :: *'b*::*topological_space set*
**assumes** *A*: *A* ∈ *sets borel* **and** *B*: *B* ∈ *sets borel*
**shows** *A* × *B* ∈ *sets borel*

**end**

## 6.7 Finite Product Measure

**theory** *Finite_Product_Measure*
**imports** *Binary_Product_Measure Function_Topology*
**begin**

### 6.7.1 Finite product spaces

**definition** *prod_emb* **where**
*prod_emb I M K X* = ($\lambda x$. *restrict x K*) −‘ $X \cap (\Pi_E\ i{\in}I.$ *space* (*M i*))

**definition** *PiM* :: $'i$ *set* ⇒ ($'i$ ⇒ $'a$ *measure*) ⇒ ($'i$ ⇒ $'a$) *measure* **where**
*PiM I M* = *extend_measure* ($\Pi_E\ i{\in}I.$ *space* (*M i*))
{(*J*, *X*). (*J* ≠ {} ∨ *I* = {}) ∧ *finite J* ∧ *J* ⊆ *I* ∧ *X* ∈ (Π *j*∈*J*. *sets* (*M j*))}
($\lambda$(*J*, *X*). *prod_emb I M J* ($\Pi_E\ j{\in}J.$ *X j*))
($\lambda$(*J*, *X*). ∏*j*∈*J* ∪ {*i*∈*I*. *emeasure* (*M i*) (*space* (*M i*)) ≠ *1*}. *if j* ∈ *J then emeasure* (*M j*) (*X j*) *else emeasure* (*M j*) (*space* (*M j*)))

**definition** *prod_algebra* :: $'i$ *set* ⇒ ($'i$ ⇒ $'a$ *measure*) ⇒ ($'i$ ⇒ $'a$) *set set* **where**
*prod_algebra I M* = ($\lambda$(*J*, *X*). *prod_emb I M J* ($\Pi_E\ j{\in}J.$ *X j*)) ‘
{(*J*, *X*). (*J* ≠ {} ∨ *I* = {}) ∧ *finite J* ∧ *J* ⊆ *I* ∧ *X* ∈ (Π *j*∈*J*. *sets* (*M j*))}

**proposition** *prod_algebra_mono*:
**assumes** *space*: ⋀*i*. *i* ∈ *I* ⟹ *space* (*E i*) = *space* (*F i*)
**assumes** *sets*: ⋀*i*. *i* ∈ *I* ⟹ *sets* (*E i*) ⊆ *sets* (*F i*)
**shows** *prod_algebra I E* ⊆ *prod_algebra I F*
**proposition** *prod_algebra_cong*:
**assumes** *I* = *J* **and** *sets*: (⋀*i*. *i* ∈ *I* ⟹ *sets* (*M i*) = *sets* (*N i*))
**shows** *prod_algebra I M* = *prod_algebra J N*

**proposition** *sets_PiM_single*: *sets* (*PiM I M*) =
*sigma_sets* ($\Pi_E\ i{\in}I.$ *space* (*M i*)) {{*f*∈$\Pi_E\ i{\in}I.$ *space* (*M i*). *f i* ∈ *A*} | *i A*. *i* ∈ *I* ∧ *A* ∈ *sets* (*M i*)}
(**is** _ = *sigma_sets ?Ω ?R*)

**proposition** *sets_PiM_sigma*:
**assumes** Ω*_cover*: ⋀*i*. *i* ∈ *I* ⟹ ∃*S*⊆*E i*. *countable S* ∧ Ω *i* = ⋃*S*
**assumes** *E*: ⋀*i*. *i* ∈ *I* ⟹ *E i* ⊆ *Pow* (Ω *i*)
**assumes** *J*: ⋀*j*. *j* ∈ *J* ⟹ *finite j* ⋃*J* = *I*
**defines** *P* ≡ {{*f*∈($\Pi_E\ i{\in}I.$ Ω *i*). ∀*i*∈*j*. *f i* ∈ *A i*} | *A j*. *j* ∈ *J* ∧ *A* ∈ *Pi j E*}
**shows** *sets* ($\Pi_M\ i{\in}I.$ *sigma* (Ω *i*) (*E i*)) = *sets* (*sigma* ($\Pi_E\ i{\in}I.$ Ω *i*) *P*)

**proposition** *measurable_PiM*:
**assumes** *space*: *f* ∈ *space N* → ($\Pi_E\ i{\in}I.$ *space* (*M i*))
**assumes** *sets*: ⋀*X J*. *J* ≠ {} ∨ *I* = {} ⟹ *finite J* ⟹ *J* ⊆ *I* ⟹ (⋀*i*. *i* ∈ *J*

$\Longrightarrow X\ i \in sets\ (M\ i)) \Longrightarrow$
$f -`\ prod\_emb\ I\ M\ J\ (Pi_E\ J\ X) \cap space\ N \in sets\ N$
**shows** $f \in measurable\ N\ (PiM\ I\ M)$

**proposition** *measurable_fun_upd*:
**assumes** $I$: $I = J \cup \{i\}$
**assumes** $f[measurable]$: $f \in measurable\ N\ (PiM\ J\ M)$
**assumes** $h[measurable]$: $h \in measurable\ N\ (M\ i)$
**shows** $(\lambda x.\ (f\ x)\ (i := h\ x)) \in measurable\ N\ (PiM\ I\ M)$

**proposition** *measure_eqI_PiM_finite*:
**assumes** [*simp*]: $finite\ I\ sets\ P = PiM\ I\ M\ sets\ Q = PiM\ I\ M$
**assumes** *eq*: $\bigwedge A.\ (\bigwedge i.\ i \in I \Longrightarrow A\ i \in sets\ (M\ i)) \Longrightarrow P\ (Pi_E\ I\ A) = Q\ (Pi_E\ I\ A)$
**assumes** $A$: $range\ A \subseteq prod\_algebra\ I\ M\ (\bigcup i.\ A\ i) = space\ (PiM\ I\ M)\ \bigwedge i::nat.\ P\ (A\ i) \neq \infty$
**shows** $P = Q$

**proposition** *measure_eqI_PiM_infinite*:
**assumes** [*simp*]: $sets\ P = PiM\ I\ M\ sets\ Q = PiM\ I\ M$
**assumes** *eq*: $\bigwedge A\ J.\ finite\ J \Longrightarrow J \subseteq I \Longrightarrow (\bigwedge i.\ i \in J \Longrightarrow A\ i \in sets\ (M\ i)) \Longrightarrow$
$P\ (prod\_emb\ I\ M\ J\ (Pi_E\ J\ A)) = Q\ (prod\_emb\ I\ M\ J\ (Pi_E\ J\ A))$
**assumes** $A$: $finite\_measure\ P$
**shows** $P = Q$

**proposition** (**in** *finite_product_sigma_finite*) *sigma_finite_pairs*:
$\exists F::'i \Rightarrow nat \Rightarrow 'a\ set.$
$(\forall i\in I.\ range\ (F\ i) \subseteq sets\ (M\ i)) \wedge$
$(\forall k.\ \forall i\in I.\ emeasure\ (M\ i)\ (F\ i\ k) \neq \infty) \wedge incseq\ (\lambda k.\ \Pi_E\ i\in I.\ F\ i\ k) \wedge$
$(\bigcup k.\ \Pi_E\ i\in I.\ F\ i\ k) = space\ (PiM\ I\ M)$

**lemma** (**in** *product_sigma_finite*) *distr_merge*:
**assumes** $IJ[simp]$: $I \cap J = \{\}$ **and** *fin*: $finite\ I\ finite\ J$
**shows** $distr\ (Pi_M\ I\ M \bigotimes_M Pi_M\ J\ M)\ (Pi_M\ (I \cup J)\ M)\ (merge\ I\ J) = Pi_M\ (I \cup J)\ M$
(**is** $?D = ?P$)

**proposition** (**in** *product_sigma_finite*) *product_nn_integral_fold*:
**assumes** $IJ$: $I \cap J = \{\}\ finite\ I\ finite\ J$
**and** $f[measurable]$: $f \in borel\_measurable\ (Pi_M\ (I \cup J)\ M)$
**shows** $integral^N\ (Pi_M\ (I \cup J)\ M)\ f =$
$(\int^+ x.\ (\int^+ y.\ f\ (merge\ I\ J\ (x, y))\ \partial(Pi_M\ J\ M))\ \partial(Pi_M\ I\ M))$

**proposition** (**in** *product_sigma_finite*) *product_nn_integral_insert*:
**assumes** $I[simp]$: $finite\ I\ i \notin I$
**and** $f$: $f \in borel\_measurable\ (Pi_M\ (insert\ i\ I)\ M)$
**shows** $integral^N\ (Pi_M\ (insert\ i\ I)\ M)\ f = (\int^+ x.\ (\int^+ y.\ f\ (x(i := y))\ \partial(M\ i))\ \partial(Pi_M\ I\ M))$

**proposition** (**in** *product_sigma_finite*) *product_nn_integral_pair*:
**assumes** [*measurable*]: $case\_prod\ f \in borel\_measurable\ (M\ x \bigotimes_M M\ y)$
**assumes** *xy*: $x \neq y$
**shows** $(\int^+\sigma.\ f\ (\sigma\ x)\ (\sigma\ y)\ \partial PiM\ \{x, y\}\ M) = (\int^+z.\ f\ (fst\ z)\ (snd\ z)\ \partial(M\ x \bigotimes_M M\ y))$

### 6.7.2 Measurability

**proposition** *sets_PiM_equal_borel*:
*sets* $(Pi_M\ UNIV\ (\lambda i::('a::countable).\ borel::('b::second\_countable\_topology\ measure))) = sets\ borel$

**end**

## 6.8 Caratheodory Extension Theorem

**theory** *Caratheodory*
**imports** *Measure_Space*
**begin**

### 6.8.1 Characterizations of Measures

**definition** *outer_measure_space* **where**
$outer\_measure\_space\ M\ f \longleftrightarrow positive\ M\ f \wedge increasing\ M\ f \wedge countably\_subadditive\ M\ f$

#### Lambda Systems

**definition** *lambda_system* :: $'a\ set \Rightarrow 'a\ set\ set \Rightarrow ('a\ set \Rightarrow ennreal) \Rightarrow 'a\ set\ set$
**where**
$lambda\_system\ \Omega\ M\ f = \{l \in M.\ \forall x \in M.\ f\ (l \cap x) + f\ ((\Omega - l) \cap x) = f\ x\}$

**proposition** (**in** *sigma_algebra*) *lambda_system_caratheodory*:
**assumes** *oms*: $outer\_measure\_space\ M\ f$
**and** *A*: $range\ A \subseteq lambda\_system\ \Omega\ M\ f$
**and** *disj*: $disjoint\_family\ A$
**shows** $(\bigcup i.\ A\ i) \in lambda\_system\ \Omega\ M\ f \wedge (\sum i.\ f\ (A\ i)) = f\ (\bigcup i.\ A\ i)$

**proposition** (**in** *sigma_algebra*) *caratheodory_lemma*:
**assumes** *oms*: $outer\_measure\_space\ M\ f$
**defines** $L \equiv lambda\_system\ \Omega\ M\ f$
**shows** $measure\_space\ \Omega\ L\ f$

**definition** *outer_measure* :: $'a\ set\ set \Rightarrow ('a\ set \Rightarrow ennreal) \Rightarrow 'a\ set \Rightarrow ennreal$
**where**

*outer_measure M f X* =
(*INF A*∈{*A*. *range A* ⊆ *M* ∧ *disjoint_family A* ∧ *X* ⊆ (⋃*i*. *A i*)}. ∑*i*. *f* (*A* *i*))

### 6.8.2 Caratheodory's theorem

**theorem** (**in** *ring_of_sets*) *caratheodory′*:
**assumes** *posf*: *positive M f* **and** *ca*: *countably_additive M f*
**shows** ∃*μ* :: ′*a set* ⇒ *ennreal*. (∀*s* ∈ *M*. *μ s* = *f s*) ∧ *measure_space* Ω (*sigma_sets* Ω *M*) *μ*

### 6.8.3 Volumes

**definition** *volume* :: ′*a set set* ⇒ (′*a set* ⇒ *ennreal*) ⇒ *bool* **where**
*volume M f* ⟷
(*f* {} = *0*) ∧ (∀*a*∈*M*. *0* ≤ *f a*) ∧
(∀*C*⊆*M*. *disjoint C* ⟶ *finite C* ⟶ ⋃*C* ∈ *M* ⟶ *f* (⋃*C*) = (∑*c*∈*C*. *f c*))

**proposition** *volume_finite_additive*:
**assumes** *volume M f*
**assumes** *A*: ⋀*i*. *i* ∈ *I* ⟹ *A i* ∈ *M disjoint_family_on A I finite I* ⋃(*A* ‘ *I*) ∈ *M*
**shows** *f* (⋃(*A* ‘ *I*)) = (∑*i*∈*I*. *f* (*A i*))

**proposition** (**in** *semiring_of_sets*) *extend_volume*:
**assumes** *volume M μ*
**shows** ∃*μ′*. *volume generated_ring μ′* ∧ (∀*a*∈*M*. *μ′ a* = *μ a*)

#### Caratheodory on semirings

**theorem** (**in** *semiring_of_sets*) *caratheodory*:
**assumes** *pos*: *positive M μ* **and** *ca*: *countably_additive M μ*
**shows** ∃*μ′* :: ′*a set* ⇒ *ennreal*. (∀*s* ∈ *M*. *μ′ s* = *μ s*) ∧ *measure_space* Ω (*sigma_sets* Ω *M*) *μ′*

**proposition** *extend_measure_caratheodory_pair*:
**fixes** *G* :: ′*i* ⇒ ′*j* ⇒ ′*a set*
**assumes** *M*: *M* = *extend_measure* Ω {(*a*, *b*). *P a b*} (λ(*a*, *b*). *G a b*) (λ(*a*, *b*). *μ a b*)
**assumes** *P i j*
**assumes** *semiring*: *semiring_of_sets* Ω {*G a b* | *a b*. *P a b*}
**assumes** *empty*: ⋀*i j*. *P i j* ⟹ *G i j* = {} ⟹ *μ i j* = *0*
**assumes** *inj*: ⋀*i j k l*. *P i j* ⟹ *P k l* ⟹ *G i j* = *G k l* ⟹ *μ i j* = *μ k l*
**assumes** *nonneg*: ⋀*i j*. *P i j* ⟹ *0* ≤ *μ i j*
**assumes** *add*: ⋀*A*::*nat* ⇒ ′*i*. ⋀*B*::*nat* ⇒ ′*j*. ⋀*j k*.
(⋀*n*. *P* (*A n*) (*B n*)) ⟹ *P j k* ⟹ *disjoint_family* (λ*n*. *G* (*A n*) (*B n*)) ⟹
(⋃*i*. *G* (*A i*) (*B i*)) = *G j k* ⟹ (∑*n*. *μ* (*A n*) (*B n*)) = *μ j k*

**shows** *emeasure M* $(G\ i\ j) = \mu\ i\ j$

**end**

## 6.9 Bochner Integration for Vector-Valued Functions

**theory** *Bochner_Integration*
**imports** *Finite_Product_Measure*
**beginproposition** *borel_measurable_implies_sequence_metric*:
**fixes** $f$ :: $'a \Rightarrow 'b$ :: $\{metric\_space, second\_countable\_topology\}$
**assumes** [*measurable*]: $f \in$ *borel_measurable M*
**shows** $\exists F.\ (\forall i.\ simple\_function\ M\ (F\ i)) \land (\forall x{\in}space\ M.\ (\lambda i.\ F\ i\ x) \longrightarrow f\ x) \land$
$(\forall i.\ \forall x{\in}space\ M.\ dist\ (F\ i\ x)\ z \le 2 * dist\ (f\ x)\ z)$

**definition** *simple_bochner_integral* :: $'a\ measure \Rightarrow ('a \Rightarrow 'b{::}real\_vector) \Rightarrow 'b$ **where**
$simple\_bochner\_integral\ M\ f = (\sum y{\in}f`space\ M.\ measure\ M\ \{x{\in}space\ M.\ f\ x = y\} *_R\ y)$

**proposition** *simple_bochner_integral_partition*:
**assumes** $f$: *simple_bochner_integrable M f* **and** $g$: *simple_function M g*
**assumes** *sub*: $\bigwedge x\ y.\ x \in space\ M \Longrightarrow y \in space\ M \Longrightarrow g\ x = g\ y \Longrightarrow f\ x = f\ y$
**assumes** $v$: $\bigwedge x.\ x \in space\ M \Longrightarrow f\ x = v\ (g\ x)$
**shows** $simple\_bochner\_integral\ M\ f = (\sum y{\in}g\ `\ space\ M.\ measure\ M\ \{x{\in}space\ M.\ g\ x = y\} *_R\ v\ y)$
(**is** _ = *?r*)

**proposition** *has_bochner_integral_implies_finite_norm*:
$has\_bochner\_integral\ M\ f\ x \Longrightarrow (\int^+ x.\ norm\ (f\ x)\ \partial M) < \infty$

**proposition** *has_bochner_integral_norm_bound*:
**assumes** $i$: *has_bochner_integral M f x*
**shows** $norm\ x \le (\int^+ x.\ norm\ (f\ x)\ \partial M)$

**definition** *lebesgue_integral* ($integral^L$) **where**
$integral^L\ M\ f = (if\ \exists x.\ has\_bochner\_integral\ M\ f\ x\ then\ THE\ x.\ has\_bochner\_integral\ M\ f\ x\ else\ 0)$

**proposition** *nn_integral_dominated_convergence_norm*:
**fixes** $u'$ :: $\_ \Rightarrow \_{::}\{real\_normed\_vector, second\_countable\_topology\}$
**assumes** [*measurable*]:
$\bigwedge i.\ u\ i \in borel\_measurable\ M\ u' \in borel\_measurable\ M\ w \in borel\_measurable\ M$
**and** *bound*: $\bigwedge j.\ AE\ x\ in\ M.\ norm\ (u\ j\ x) \le w\ x$
**and** $w$: $(\int^+ x.\ w\ x\ \partial M) < \infty$
**and** $u'$: $AE\ x\ in\ M.\ (\lambda i.\ u\ i\ x) \longrightarrow u'\ x$

**shows** $(\lambda i.\ (\int^+ x.\ norm\ (u'\ x - u\ i\ x)\ \partial M)) \longrightarrow 0$

**proposition** *integrableI_bounded*:
**fixes** $f :: 'a \Rightarrow 'b$::{*banach*, *second_countable_topology*}
**assumes** $f$[*measurable*]: $f \in borel\_measurable\ M$ **and** *fin*: $(\int^+ x.\ norm\ (f\ x)\ \partial M) < \infty$
**shows** *integrable M f*

**proposition** *nn_integral_eq_integral*:
**assumes** $f$: *integrable M f*
**assumes** *nonneg*: $AE\ x\ in\ M.\ 0 \le f\ x$
**shows** $(\int^+ x.\ f\ x\ \partial M) = integral^L\ M\ f$

**proposition** *integral_norm_bound* [*simp*]:
**fixes** $f :: \_ \Rightarrow 'a :: \{banach,\ second\_countable\_topology\}$
**shows** $norm\ (integral^L\ M\ f) \le (\int x.\ norm\ (f\ x)\ \partial M)$

**proposition** *integral_abs_bound* [*simp*]:
**fixes** $f :: 'a \Rightarrow real$ **shows** $abs\ (\int x.\ f\ x\ \partial M) \le (\int x.\ |f\ x|\ \partial M)$

**proposition** *integrable_induct*[*consumes 1*, *case_names base add lim*, *induct pred*: *integrable*]:
**fixes** $f :: 'a \Rightarrow 'b$::{*banach*, *second_countable_topology*}
**assumes** *integrable M f*
**assumes** *base*: $\bigwedge A\ c.\ A \in sets\ M \Longrightarrow emeasure\ M\ A < \infty \Longrightarrow P\ (\lambda x.\ indicator\ A\ x *_R c)$
**assumes** *add*: $\bigwedge f\ g.\ integrable\ M\ f \Longrightarrow P\ f \Longrightarrow integrable\ M\ g \Longrightarrow P\ g \Longrightarrow P\ (\lambda x.\ f\ x + g\ x)$
**assumes** *lim*: $\bigwedge f\ s.\ (\bigwedge i.\ integrable\ M\ (s\ i)) \Longrightarrow (\bigwedge i.\ P\ (s\ i)) \Longrightarrow$
$(\bigwedge x.\ x \in space\ M \Longrightarrow (\lambda i.\ s\ i\ x) \longrightarrow f\ x) \Longrightarrow$
$(\bigwedge i\ x.\ x \in space\ M \Longrightarrow norm\ (s\ i\ x) \le 2 * norm\ (f\ x)) \Longrightarrow integrable\ M\ f \Longrightarrow P\ f$
**shows** *P f*

**theorem** *integral_Markov_inequality*:
**assumes** [*measurable*]: *integrable M u* **and** $AE\ x\ in\ M.\ 0 \le u\ x\ 0 < (c::real)$
**shows** $(emeasure\ M)\ \{x \in space\ M.\ u\ x \ge c\} \le (1/c) * (\int x.\ u\ x\ \partial M)$

**theorem** *integral_Markov_inequality_measure*:
**assumes** [*measurable*]: *integrable M u* **and** $A \in sets\ M$ **and** $AE\ x\ in\ M.\ 0 \le u\ x\ 0 < (c::real)$
**shows** $measure\ M\ \{x \in space\ M.\ u\ x \ge c\} \le (\int x.\ u\ x\ \partial M)\ /\ c$

**theorem** (**in** *finite_measure*) *second_moment_method*:
**assumes** [*measurable*]: $f \in M \rightarrow_M borel$
**assumes** $integrable\ M\ (\lambda x.\ f\ x\ \hat{}\ 2)$
**defines** $\mu \equiv lebesgue\_integral\ M\ f$
**assumes** $a > 0$

**shows** *measure M* $\{x \in space\ M.\ |f\ x| \geq a\} \leq lebesgue\_integral\ M\ (\lambda x.\ f\ x\ \hat{}\ 2) / a^2$
**proof** –
  **have** *integrable*: *integrable M f*
    **using** *assms* **by** (*blast dest*: *square_integrable_imp_integrable*)
  **have** $\{x \in space\ M.\ |f\ x| \geq a\} = \{x \in space\ M.\ f\ x\ \hat{}\ 2 \geq a^2\}$
    **using** ‹$a > 0$› **by** (*simp flip*: *abs_le_square_iff*)
  **hence** *measure M* $\{x \in space\ M.\ |f\ x| \geq a\} = measure\ M\ \{x \in space\ M.\ f\ x\ \hat{}\ 2 \geq a^2\}$
    **by** *simp*
  **also have** ... $\leq lebesgue\_integral\ M\ (\lambda x.\ f\ x\ \hat{}\ 2) / a^2$
    **using** *assms* **by** (*intro integral_Markov_inequality_measure*) *auto*
  **finally show** *?thesis* **.**
**qed**

**proposition** *tendsto_L1_int*:
  **fixes** $u :: \_ \Rightarrow \_ \Rightarrow 'b::\{banach, second\_countable\_topology\}$
  **assumes** [*measurable*]: $\bigwedge n.\ integrable\ M\ (u\ n)\ integrable\ M\ f$
      **and** $((\lambda n.\ (\int^+ x.\ norm(u\ n\ x - f\ x)\ \partial M)) \longrightarrow 0)\ F$
  **shows** $((\lambda n.\ (\int x.\ u\ n\ x\ \partial M)) \longrightarrow (\int x.\ f\ x\ \partial M))\ F$
**proposition** *tendsto_L1_AE_subseq*:
  **fixes** $u :: nat \Rightarrow 'a \Rightarrow 'b::\{banach, second\_countable\_topology\}$
  **assumes** [*measurable*]: $\bigwedge n.\ integrable\ M\ (u\ n)$
    **and** $(\lambda n.\ (\int x.\ norm(u\ n\ x)\ \partial M)) \longlonglongrightarrow 0$
  **shows** $\exists r::nat \Rightarrow nat.\ strict\_mono\ r \wedge (AE\ x\ in\ M.\ (\lambda n.\ u\ (r\ n)\ x) \longlonglongrightarrow 0)$

### 6.9.1 Restricted measure spaces

### 6.9.2 Measure spaces with an associated density

### 6.9.3 Distributions

### 6.9.4 Lebesgue integration on *count_space*

### 6.9.5 Point measure

**proposition** *integrable_point_measure_finite*:
  **fixes** $g :: 'a \Rightarrow 'b::\{banach, second\_countable\_topology\}$ **and** $f :: 'a \Rightarrow real$
  **assumes** *finite A*
  **shows** *integrable* (*point_measure A f*) *g*

### 6.9.6 Lebesgue integration on *null_measure*

### 6.9.7 Legacy lemmas for the real-valued Lebesgue integral

**theorem** *real_lebesgue_integral_def*:
  **assumes** *f*[*measurable*]: *integrable M f*

**shows** $integral^L\ M\ f = enn2real\ (\int^+ x.\ f\ x\ \partial M) - enn2real\ (\int^+ x.\ ennreal\ (-\ f\ x)\ \partial M)$

**theorem** *real_integrable_def*:
  $integrable\ M\ f \longleftrightarrow f \in borel\_measurable\ M\ \wedge$
    $(\int^+ x.\ ennreal\ (f\ x)\ \partial M) \neq \infty \wedge (\int^+ x.\ ennreal\ (-\ f\ x)\ \partial M) \neq \infty$

### 6.9.8 Product measure

**proposition** (**in** *sigma_finite_measure*) *borel_measurable_lebesgue_integral*[*measurable* (*raw*)]:
  **fixes** $f :: \_ \Rightarrow \_ \Rightarrow \_::\{banach,\ second\_countable\_topology\}$
  **assumes** *f*[*measurable*]: $case\_prod\ f \in borel\_measurable\ (N \bigotimes_M M)$
  **shows** $(\lambda x.\ \int y.\ f\ x\ y\ \partial M) \in borel\_measurable\ N$

**theorem** (**in** *pair_sigma_finite*) *Fubini_integrable*:
  **fixes** $f :: \_ \Rightarrow \_::\{banach,\ second\_countable\_topology\}$
  **assumes** *f*[*measurable*]: $f \in borel\_measurable\ (M1 \bigotimes_M M2)$
    **and** *integ1*: $integrable\ M1\ (\lambda x.\ \int y.\ norm\ (f\ (x,\ y))\ \partial M2)$
    **and** *integ2*: $AE\ x\ in\ M1.\ integrable\ M2\ (\lambda y.\ f\ (x,\ y))$
  **shows** $integrable\ (M1 \bigotimes_M M2)\ f$

**proposition** (**in** *pair_sigma_finite*) *integral_fst'*:
  **fixes** $f :: \_ \Rightarrow \_::\{banach,\ second\_countable\_topology\}$
  **assumes** *f*: $integrable\ (M1 \bigotimes_M M2)\ f$
  **shows** $(\int x.\ (\int y.\ f\ (x,\ y)\ \partial M2)\ \partial M1) = integral^L\ (M1 \bigotimes_M M2)\ f$

**proposition** (**in** *pair_sigma_finite*) *Fubini_integral*:
  **fixes** $f :: \_ \Rightarrow \_ \Rightarrow \_ :: \{banach,\ second\_countable\_topology\}$
  **assumes** *f*: $integrable\ (M1 \bigotimes_M M2)\ (case\_prod\ f)$
  **shows** $(\int y.\ (\int x.\ f\ x\ y\ \partial M1)\ \partial M2) = (\int x.\ (\int y.\ f\ x\ y\ \partial M2)\ \partial M1)$

**end**

## 6.10 Complete Measures

**theory** *Complete_Measure*
  **imports** *Bochner_Integration*
**begin**

**locale** *complete_measure* =
  **fixes** $M :: 'a\ measure$
  **assumes** *complete*: $\bigwedge A\ B.\ B \subseteq A \Longrightarrow A \in null\_sets\ M \Longrightarrow B \in sets\ M$

**definition**
  $split\_completion\ M\ A\ p = (if\ A \in sets\ M\ then\ p = (A,\ \{\})\ else$
   $\exists N'.\ A = fst\ p \cup snd\ p \wedge fst\ p \cap snd\ p = \{\} \wedge fst\ p \in sets\ M \wedge snd\ p \subseteq N' \wedge$

$N' \in$ *null_sets M*)

**definition**
*main_part M A* = *fst* (*Eps* (*split_completion M A*))

**definition**
*null_part M A* = *snd* (*Eps* (*split_completion M A*))

**definition** *completion* :: *'a measure* ⇒ *'a measure* **where**
*completion M* = *measure_of* (*space M*) { *S* ∪ *N* |*S N N'*. *S* ∈ *sets M* ∧ *N'* ∈ *null_sets M* ∧ *N* ⊆ *N'* }
(*emeasure M* ∘ *main_part M*)

**lemma** *sets_completion*:
*sets* (*completion M*) = { *S* ∪ *N* |*S N N'*. *S* ∈ *sets M* ∧ *N'* ∈ *null_sets M* ∧ *N* ⊆ *N'* }

**lemma** *measurable_completion*: $f \in M \to_M N \Longrightarrow f \in$ *completion* $M \to_M N$

**lemma** *split_completion*:
**assumes** *A* ∈ *sets* (*completion M*)
**shows** *split_completion M A* (*main_part M A*, *null_part M A*)

**lemma** *emeasure_completion*[*simp*]:
**assumes** *S*: *S* ∈ *sets* (*completion M*)
**shows** *emeasure* (*completion M*) *S* = *emeasure M* (*main_part M S*)

**lemma** *completion_ex_borel_measurable*:
**fixes** *g* :: *'a* ⇒ *ennreal*
**assumes** *g*: *g* ∈ *borel_measurable* (*completion M*)
**shows** ∃ *g'*∈*borel_measurable M*. (*AE x in M*. *g x* = *g' x*)

**locale** *semifinite_measure* =
**fixes** *M* :: *'a measure*
**assumes** *semifinite*:
⋀*A*. *A* ∈ *sets M* ⟹ *emeasure M A* = ∞ ⟹ ∃ *B*∈*sets M*. *B* ⊆ *A* ∧ *emeasure M B* < ∞

**locale** *locally_determined_measure* = *semifinite_measure* +
**assumes** *locally_determined*:
⋀*A*. *A* ⊆ *space M* ⟹ (⋀*B*. *B* ∈ *sets M* ⟹ *emeasure M B* < ∞ ⟹ *A* ∩ *B* ∈ *sets M*) ⟹ *A* ∈ *sets M*

**locale** *cld_measure* =
*complete_measure M* + *locally_determined_measure M* **for** *M* :: *'a measure*

**definition** *outer_measure_of* :: *'a measure* ⇒ *'a set* ⇒ *ennreal*
**where** *outer_measure_of M A* = (*INF B* ∈ {*B*∈*sets M*. *A* ⊆ *B*}. *emeasure M B*)

**definition** *measurable_envelope* :: *'a measure* ⇒ *'a set* ⇒ *'a set* ⇒ *bool*
**where** *measurable_envelope M A E* ⟷
($A \subseteq E \wedge E \in sets\ M \wedge (\forall F{\in}sets\ M.\ emeasure\ M\ (F \cap E) = outer\_measure\_of\ M\ (F \cap A))$)

**lemma** *measurable_envelope_eq2*:
**assumes** $A \subseteq E$ $E \in sets\ M$ $emeasure\ M\ E < \infty$
**shows** $measurable\_envelope\ M\ A\ E \longleftrightarrow (emeasure\ M\ E = outer\_measure\_of\ M\ A)$

**proposition** (**in** *complete_measure*) *fmeasurable_inner_outer*:
$S \in fmeasurable\ M \longleftrightarrow$
$(\forall e{>}0.\ \exists T{\in}fmeasurable\ M.\ \exists U{\in}fmeasurable\ M.\ T \subseteq S \wedge S \subseteq U \wedge |measure\ M\ T - measure\ M\ U| < e)$
(**is** _ ⟷ *?approx*)

**end**

## 6.11 Regularity of Measures

**theory** *Regularity*
**imports** *Measure_Space Borel_Space*
**begin**

**theorem**
**fixes** *M*::*'a*::{*second_countable_topology*, *complete_space*} *measure*
**assumes** *sb*: $sets\ M = sets\ borel$
**assumes** $emeasure\ M\ (space\ M) \neq \infty$
**assumes** $B \in sets\ borel$
**shows** *inner_regular*: $emeasure\ M\ B =$
$(SUP\ K \in \{K.\ K \subseteq B \wedge compact\ K\}.\ emeasure\ M\ K)$ (**is** *?inner B*)
**and** *outer_regular*: $emeasure\ M\ B =$
$(INF\ U \in \{U.\ B \subseteq U \wedge open\ U\}.\ emeasure\ M\ U)$ (**is** *?outer B*)

**end**

## 6.12 Lebesgue Measure

**theory** *Lebesgue_Measure*
**imports**
*Finite_Product_Measure*
*Caratheodory*
*Complete_Measure*
*Summation_Tests*
*Regularity*
**begin**

### 6.12.1 Measures defined by monotonous functions

**definition** *interval_measure* :: (*real* ⇒ *real*) ⇒ *real measure* **where**
*interval_measure F* =
*extend_measure UNIV* {(*a*, *b*). $a \leq b$} ($\lambda$(*a*, *b*). {*a*<..*b*}) ($\lambda$(*a*, *b*). *ennreal* (*F b* − *F a*))

**lemma** *emeasure_interval_measure_Ioc*:
**assumes** $a \leq b$
**assumes** *mono_F*: $\bigwedge x\ y.\ x \leq y \Longrightarrow F\ x \leq F\ y$
**assumes** *right_cont_F* : $\bigwedge a.$ *continuous* (*at_right a*) *F*
**shows** *emeasure* (*interval_measure F*) {*a*<..*b*} = *F b* − *F a*

**lemma** *sets_interval_measure* [*simp*, *measurable_cong*]:
*sets* (*interval_measure F*) = *sets borel*

**lemma** *sigma_finite_interval_measure*:
**assumes** *mono_F*: $\bigwedge x\ y.\ x \leq y \Longrightarrow F\ x \leq F\ y$
**assumes** *right_cont_F* : $\bigwedge a.$ *continuous* (*at_right a*) *F*
**shows** *sigma_finite_measure* (*interval_measure F*)

### 6.12.2 Lebesgue-Borel measure

**definition** *lborel* :: (*'a* :: *euclidean_space*) *measure* **where**
*lborel* = *distr* ($\Pi_M$ *b*∈*Basis*. *interval_measure* ($\lambda x.\ x$)) *borel* ($\lambda f.$ $\sum$ *b*∈*Basis*. *f b* $*_R$ *b*)

**abbreviation** *lebesgue* :: *'a*::*euclidean_space measure*
**where** *lebesgue* ≡ *completion lborel*

**abbreviation** *lebesgue_on* :: *'a set* ⇒ *'a*::*euclidean_space measure*
**where** *lebesgue_on* Ω ≡ *restrict_space* (*completion lborel*) Ω

### 6.12.3 Borel measurability

**lemma** *emeasure_lborel_cbox*[*simp*]:
**assumes** [*simp*]: $\bigwedge b.\ b \in$ *Basis* $\Longrightarrow l \cdot b \leq u \cdot b$
**shows** *emeasure lborel* (*cbox l u*) = ($\prod$ *b*∈*Basis*. $(u - l) \cdot b$)

### 6.12.4 Affine transformation on the Lebesgue-Borel

**lemma** *lborel_eqI*:
**fixes** *M* :: *'a*::*euclidean_space measure*
**assumes** *emeasure_eq*: $\bigwedge l\ u.$ ($\bigwedge b.\ b \in$ *Basis* $\Longrightarrow l \cdot b \leq u \cdot b$) $\Longrightarrow$ *emeasure M* (*box l u*) = ($\prod$ *b*∈*Basis*. $(u - l) \cdot b$)
**assumes** *sets_eq*: *sets M* = *sets borel*

**shows** *lborel* = *M*

**lemma** *lborel_affine_euclidean*:
**fixes** $c$ :: *'a::euclidean_space* ⇒ *real* **and** $t$
**defines** $T\ x \equiv t + (\sum j\in Basis.\ (c\ j * (x \cdot j)) *_R j)$
**assumes** *c*: $\bigwedge j.\ j \in Basis \Longrightarrow c\ j \neq 0$
**shows** *lborel* = *density* (*distr lborel borel T*) ($\lambda$_. ($\prod j\in Basis.\ |c\ j|$)) (**is** _ = *?D*)

**lemma** *lborel_integral_real_affine*:
**fixes** $f$ :: *real* ⇒ *'a* :: {*banach*, *second_countable_topology*} **and** $c$ :: *real*
**assumes** *c*: $c \neq 0$ **shows** ($\int x.\ f\ x\ \partial\ lborel$) = $|c| *_R (\int x.\ f\ (t + c * x)\ \partial lborel)$

**corollary** *lebesgue_real_affine*:
$c \neq 0 \Longrightarrow$ *lebesgue* = *density* (*distr lebesgue lebesgue* ($\lambda x.\ t + c * x$)) ($\lambda$_. *ennreal* (*abs* $c$))

**lemma** *lborel_prod*:
*lborel* $\bigotimes_M$ *lborel* = (*lborel* :: (*'a::euclidean_space* × *'b::euclidean_space*) *measure*)

### 6.12.5 Lebesgue measurable sets

**abbreviation** *lmeasurable* :: *'a::euclidean_space set set*
**where**
*lmeasurable* ≡ *fmeasurable lebesgue*

**lemma** *lmeasurable_iff_integrable*:
$S \in$ *lmeasurable* ⟷ *integrable lebesgue* (*indicator* $S$ :: *'a::euclidean_space* ⇒ *real*)

### 6.12.6 A nice lemma for negligibility proofs

**proposition** *starlike_negligible_bounded_gmeasurable*:
**fixes** $S$ :: *'a* :: *euclidean_space set*
**assumes** *S*: $S \in$ *sets lebesgue* **and** *bounded* $S$
**and** *eq1*: $\bigwedge c\ x.\ [\![(c *_R x) \in S;\ 0 \leq c;\ x \in S]\!] \Longrightarrow c = 1$
**shows** $S \in$ *null_sets lebesgue*

**corollary** *starlike_negligible_compact*:
*compact* $S \Longrightarrow (\bigwedge c\ x.\ [\![(c *_R x) \in S;\ 0 \leq c;\ x \in S]\!] \Longrightarrow c = 1) \Longrightarrow S \in$ *null_sets lebesgue*

**proposition** *outer_regular_lborel_le*:
**assumes** *B*[*measurable*]: $B \in$ *sets borel* **and** $0 <$ (*e::real*)
**obtains** $U$ **where** *open* $U$ $B \subseteq U$ **and** *emeasure lborel* $(U - B) \leq e$

**lemma** *outer_regular_lborel*:
**assumes** *B*: $B \in$ *sets borel* **and** $0 < (e::real)$
**obtains** *U* **where** *open U* $B \subseteq U$ *emeasure lborel* $(U - B) < e$

### 6.12.7 *F_sigma* and *G_delta* sets.

**inductive** *fsigma* :: *'a::topological_space set* ⇒ *bool* **where**
$(\bigwedge n::nat.\ closed\ (F\ n)) \Longrightarrow fsigma\ (\bigcup (F\ `\ UNIV))$

**inductive** *gdelta* :: *'a::topological_space set* ⇒ *bool* **where**
$(\bigwedge n::nat.\ open\ (F\ n)) \Longrightarrow gdelta\ (\bigcap (F\ `\ UNIV))$

**end**

## 6.13 Tagged Divisions for Henstock-Kurzweil Integration

**theory** *Tagged_Division*
**imports** *Topology_Euclidean_Space*
**begin**

### 6.13.1 Some useful lemmas about intervals

### 6.13.2 Bounds on intervals where they exist

**definition** *interval_upperbound* :: *('a::euclidean_space) set* ⇒ *'a*
**where** *interval_upperbound s* = $(\sum i \in Basis.\ (SUP\ x \in s.\ x \cdot i) *_R i)$

**definition** *interval_lowerbound* :: *('a::euclidean_space) set* ⇒ *'a*
**where** *interval_lowerbound s* = $(\sum i \in Basis.\ (INF\ x \in s.\ x \cdot i) *_R i)$

### 6.13.3 The notion of a gauge — simply an open set containing the point

**definition** *gauge* $\gamma \longleftrightarrow (\forall x.\ x \in \gamma\ x \wedge open\ (\gamma\ x))$

### 6.13.4 Attempt a systematic general set of "offset" results for components

### 6.13.5 Divisions

**definition** *division_of* (**infixl** *division'_of 40*)
**where**
*s division_of i* ⟷
*finite s* ∧

$(\forall K \in s.\ K \subseteq i \wedge K \neq \{\} \wedge (\exists a\ b.\ K = cbox\ a\ b)) \wedge$
$(\forall K1 \in s.\ \forall K2 \in s.\ K1 \neq K2 \longrightarrow interior(K1) \cap interior(K2) = \{\}) \wedge$
$(\bigcup s = i)$

**proposition** *partial_division_extend_interval*:
**assumes** $p$ *division_of* $(\bigcup p)$ $(\bigcup p) \subseteq cbox\ a\ b$
**obtains** $q$ **where** $p \subseteq q$ $q$ *division_of* $cbox\ a\ (b::'a::euclidean\_space)$

**proposition** *division_union_intervals_exists*:
**assumes** $cbox\ a\ b \neq \{\}$
**obtains** $p$ **where** $(insert\ (cbox\ a\ b)\ p)$ *division_of* $(cbox\ a\ b \cup cbox\ c\ d)$

### 6.13.6 Tagged (partial) divisions

**definition** *tagged_partial_division_of* (**infixr** *tagged'_partial'_division'_of 40*)
**where** $s$ *tagged_partial_division_of* $i \longleftrightarrow$
*finite* $s \wedge$
$(\forall x\ K.\ (x, K) \in s \longrightarrow x \in K \wedge K \subseteq i \wedge (\exists a\ b.\ K = cbox\ a\ b)) \wedge$
$(\forall x1\ K1\ x2\ K2.\ (x1, K1) \in s \wedge (x2, K2) \in s \wedge (x1, K1) \neq (x2, K2) \longrightarrow$
$interior\ K1 \cap interior\ K2 = \{\})$

**definition** *tagged_division_of* (**infixr** *tagged'_division'_of 40*)
**where** $s$ *tagged_division_of* $i \longleftrightarrow s$ *tagged_partial_division_of* $i \wedge (\bigcup \{K.\ \exists x.\ (x,K) \in s\} = i)$

### 6.13.7 Functions closed on boxes: morphisms from boxes to monoids

**Using additivity of lifted function to encode definedness.** **definition** *lift_option* :: $('a \Rightarrow 'b \Rightarrow 'c) \Rightarrow 'a\ option \Rightarrow 'b\ option \Rightarrow 'c\ option$
**where**
$lift\_option\ f\ a'\ b' = Option.bind\ a'\ (\lambda a.\ Option.bind\ b'\ (\lambda b.\ Some\ (f\ a\ b)))$

**lemma** *comm_monoid_lift_option*:
**assumes** *comm_monoid f z*
**shows** *comm_monoid* (*lift_option f*) (*Some z*)

#### Misc

**Division points** **definition** $division\_points\ (k::('a::euclidean\_space)\ set)\ d =$
$\{(j,x).\ j \in Basis \wedge (interval\_lowerbound\ k)\cdot j < x \wedge x < (interval\_upperbound\ k)\cdot j \wedge$
$(\exists i \in d.\ (interval\_lowerbound\ i)\cdot j = x \vee (interval\_upperbound\ i)\cdot j = x)\}$

**Operative**

**proposition** *tagged_division*:
**assumes** $d$ *tagged_division_of* $(cbox\ a\ b)$
**shows** $F\ (\lambda(\_,\ l).\ g\ l)\ d = g\ (cbox\ a\ b)$

### 6.13.8 Special case of additivity we need for the FTC

### 6.13.9 Fine-ness of a partition w.r.t. a gauge

**definition** *fine* (**infixr** *fine 46*)
**where** $d$ *fine* $s \longleftrightarrow (\forall (x,k) \in s.\ k \subseteq d\ x)$

### 6.13.10 Some basic combining lemmas

### 6.13.11 General bisection principle for intervals; might be useful elsewhere

### 6.13.12 Cousin's lemma

### 6.13.13 A technical lemma about "refinement" of division

#### Covering lemma

**proposition** *covering_lemma*:
**assumes** $S \subseteq cbox\ a\ b$ $box\ a\ b \neq \{\}$ $gauge\ g$
**obtains** $\mathcal{D}$ **where**
$countable\ \mathcal{D}$ $\bigcup\mathcal{D} \subseteq cbox\ a\ b$
$\bigwedge K.\ K \in \mathcal{D} \Longrightarrow interior\ K \neq \{\} \wedge (\exists\, c\ d.\ K = cbox\ c\ d)$
$pairwise\ (\lambda A\ B.\ interior\ A \cap interior\ B = \{\})\ \mathcal{D}$
$\bigwedge K.\ K \in \mathcal{D} \Longrightarrow \exists\, x \in S \cap K.\ K \subseteq g\ x$
$\bigwedge u\ v.\ cbox\ u\ v \in \mathcal{D} \Longrightarrow \exists\, n.\ \forall\, i \in Basis.\ v \cdot i - u \cdot i = (b \cdot i - a \cdot i)\ /\ 2\hat{\ }n$
$S \subseteq \bigcup\mathcal{D}$

### 6.13.14 Division filter

**definition** *division_filter* :: $'a::euclidean\_space\ set \Rightarrow ('a \times 'a\ set)\ set\ filter$
**where** $division\_filter\ s = (INF\ g{\in}\{g.\ gauge\ g\}.\ principal\ \{p.\ p\ tagged\_division\_of\ s \wedge g\ fine\ p\})$

**proposition** *eventually_division_filter*:
$(\forall_F\ p\ in\ division\_filter\ s.\ P\ p) \longleftrightarrow$
$(\exists\, g.\ gauge\ g \wedge (\forall\, p.\ p\ tagged\_division\_of\ s \wedge g\ fine\ p \longrightarrow P\ p))$

**end**

## 6.14 Henstock-Kurzweil Gauge Integration in Many Dimensions

**theory** *Henstock_Kurzweil_Integration*
**imports**
  *Lebesgue_Measure Tagged_Division*
**begin**

### 6.14.1 Content (length, area, volume...) of an interval

### 6.14.2 Gauge integral

### 6.14.3 Basic theorems about integrals

**corollary** *integral_mult_left* [*simp*]:
  **fixes** *c*:: *'a*::{*real_normed_algebra,division_ring*}
  **shows** *integral S* (*λx. f x* ∗ *c*) = *integral S f* ∗ *c*

**corollary** *integral_mult_right* [*simp*]:
  **fixes** *c*:: *'a*::{*real_normed_field*}
  **shows** *integral S* (*λx. c* ∗ *f x*) = *c* ∗ *integral S f*

**corollary** *integral_divide* [*simp*]:
  **fixes** *z* :: *'a*::*real_normed_field*
  **shows** *integral S* (*λx. f x* / *z*) = *integral S* (*λx. f x*) / *z*

### 6.14.4 Cauchy-type criterion for integrability

**proposition** *integrable_Cauchy*:
  **fixes** *f* :: *'n*::*euclidean_space* ⇒ *'a*::{*real_normed_vector,complete_space*}
  **shows** *f integrable_on cbox a b* ⟷
    (∀ *e*>*0*. ∃γ. *gauge* γ ∧
     (∀𝒟*1* 𝒟*2*. 𝒟*1 tagged_division_of* (*cbox a b*) ∧ γ *fine* 𝒟*1* ∧
      𝒟*2 tagged_division_of* (*cbox a b*) ∧ γ *fine* 𝒟*2* ⟶
      *norm* ((∑(*x,K*)∈𝒟*1*. *content K* $*_R$ *f x*) − (∑(*x,K*)∈𝒟*2*. *content K* $*_R$
*f x*)) < *e*))
  (**is** *?l* = (∀ *e*>*0*. ∃γ. *?P e* γ))

### 6.14.5 Additivity of integral on abutting intervals

**proposition** *has_integral_split*:
  **fixes** *f* :: *'a*::*euclidean_space* ⇒ *'b*::*real_normed_vector*
  **assumes** *fi*: (*f has_integral i*) (*cbox a b* ∩ {*x. x*·*k* ≤ *c*})
    **and** *fj*: (*f has_integral j*) (*cbox a b* ∩ {*x. x*·*k* ≥ *c*})
    **and** *k*: *k* ∈ *Basis*

**shows** ($f$ *has_integral* $(i + j)$) (*cbox* $a$ $b$)

### 6.14.6 A sort of converse, integrability on subintervals

### 6.14.7 Bounds on the norm of Riemann sums and the integral itself

**corollary** *integrable_bound*:
**fixes** $f$ :: *'a::euclidean_space* ⇒ *'b::real_normed_vector*
**assumes** $0 \leq B$
**and** $f$ *integrable_on* (*cbox* $a$ $b$)
**and** $\bigwedge x.\ x \in$ *cbox* $a$ $b \Longrightarrow$ *norm* $(f\ x) \leq B$
**shows** *norm* (*integral* (*cbox* $a$ $b$) $f$) $\leq B *$ *content* (*cbox* $a$ $b$)

### 6.14.8 Similar theorems about relationship among components

### 6.14.9 Uniform limit of integrable functions is integrable

### 6.14.10 Negligible sets

**proposition** *negligible_standard_hyperplane*[*intro*]:
**fixes** $k$ :: *'a::euclidean_space*
**assumes** $k$: $k \in$ *Basis*
**shows** *negligible* $\{x.\ x \cdot k = c\}$

**corollary** *negligible_standard_hyperplane_cart*:
**fixes** $k$ :: *'a::finite*
**shows** *negligible* $\{x.\ x\$k = (0::real)\}$

**proposition** *has_integral_negligible*:
**fixes** $f$ :: *'b::euclidean_space* ⇒ *'a::real_normed_vector*
**assumes** *negs*: *negligible* $S$
**and** $\bigwedge x.\ x \in (T - S) \Longrightarrow f\ x = 0$
**shows** ($f$ *has_integral* $0$) $T$

### 6.14.11 Some other trivialities about negligible sets

### 6.14.12 Finite case of the spike theorem is quite commonly needed

**corollary** *has_integral_bound_real*:
**fixes** $f$ :: *real* ⇒ *'b::real_normed_vector*
**assumes** $0 \le B$ *finite* $S$
**and** ($f$ *has_integral* $i$) $\{a..b\}$
**and** $\bigwedge x.\ x \in \{a..b\} - S \Longrightarrow$ *norm* $(f\ x) \le B$
**shows** *norm* $i \le B *$ *content* $\{a..b\}$

### 6.14.13 In particular, the boundary of an interval is negligible

### 6.14.14 Integrability of continuous functions

### 6.14.15 Specialization of additivity to one dimension

### 6.14.16 A useful lemma allowing us to factor out the content size

### 6.14.17 Fundamental theorem of calculus

**theorem** *fundamental_theorem_of_calculus*:
**fixes** $f$ :: *real* ⇒ *'a::banach*
**assumes** $a \le b$
**and** *vecd*: $\bigwedge x.\ x \in \{a..b\} \Longrightarrow$ ($f$ *has_vector_derivative* $f'$ $x$) (*at* $x$ *within* $\{a..b\}$)
**shows** ($f'$ *has_integral* $(f\ b - f\ a)$) $\{a..b\}$

### 6.14.18 Taylor series expansion

### 6.14.19 Only need trivial subintervals if the interval itself is trivial

**proposition** *division_of_nontrivial*:
**fixes** $\mathcal{D}$ :: *'a::euclidean_space set set*
**assumes** *sdiv*: $\mathcal{D}$ *division_of* (*cbox* $a$ $b$)
**and** *cont0*: *content* (*cbox* $a$ $b$) $\neq 0$
**shows** $\{k.\ k \in \mathcal{D} \wedge$ *content* $k \neq 0\}$ *division_of* (*cbox* $a$ $b$)

#### 6.14.20 Integrability on subintervals

#### 6.14.21 Combining adjacent intervals in 1 dimension

#### 6.14.22 Reduce integrability to "local" integrability

#### 6.14.23 Second FTC or existence of antiderivative

#### 6.14.24 Combined fundamental theorem of calculus

#### 6.14.25 General "twiddling" for interval-to-interval function image

#### 6.14.26 Special case of a basic affine transformation

#### 6.14.27 Special case of stretching coordinate axes separately

#### 6.14.28 even more special cases

#### 6.14.29 Stronger form of FCT; quite a tedious proof

**theorem** *fundamental_theorem_of_calculus_interior*:
**fixes** $f :: real \Rightarrow 'a::real\_normed\_vector$
**assumes** $a \leq b$
**and** *contf*: *continuous_on* $\{a..b\}$ $f$
**and** *derf*: $\bigwedge x.\ x \in \{a <..< b\} \Longrightarrow (f\ has\_vector\_derivative\ f'\ x)\ (at\ x)$
**shows** $(f'\ has\_integral\ (f\ b - f\ a))\ \{a..b\}$

#### 6.14.30 Stronger form with finite number of exceptional points

**corollary** *fundamental_theorem_of_calculus_strong*:
**fixes** $f :: real \Rightarrow 'a::banach$
**assumes** *finite S*
**and** $a \leq b$
**and** *vec*: $\bigwedge x.\ x \in \{a..b\} - S \Longrightarrow (f\ has\_vector\_derivative\ f'(x))\ (at\ x)$
**and** *continuous_on* $\{a..b\}$ $f$
**shows** $(f'\ has\_integral\ (f\ b - f\ a))\ \{a..b\}$

**proposition** *indefinite_integral_continuous_left*:
**fixes** $f:: real \Rightarrow 'a::banach$
**assumes** *intf*: $f$ *integrable_on* $\{a..b\}$ **and** $a < c\ c \leq b\ e > 0$
**obtains** $d$ **where** $d > 0$
**and** $\forall t.\ c - d < t \wedge t \leq c \longrightarrow norm\ (integral\ \{a..c\}\ f - integral\ \{a..t\}\ f) < e$

**theorem** *integral_has_vector_derivative'*:

**fixes** $f$ :: *real* ⇒ *'b::banach*
**assumes** *continuous_on* $\{a..b\}$ $f$
**and** $x \in \{a..b\}$
**shows** $((\lambda u.$ *integral* $\{u..b\}$ $f)$ *has_vector_derivative* $- f\ x)$ (*at* $x$ *within* $\{a..b\}$)

### 6.14.31 This doesn't directly involve integration, but that gives an easy proof

### 6.14.32 Generalize a bit to any convex set

### 6.14.33 Integrating characteristic function of an interval

**corollary** *has_integral_restrict_UNIV*:
**fixes** $f$ :: *'n::euclidean_space* ⇒ *'a::banach*
**shows** $((\lambda x.$ *if* $x \in s$ *then* $f\ x$ *else* $0)$ *has_integral* $i)$ *UNIV* ⟷ ($f$ *has_integral* $i$) $s$

### 6.14.34 Integrals on set differences

**corollary** *integral_spike_set*:
**fixes** $f$ :: *'n::euclidean_space* ⇒ *'a::banach*
**assumes** *negligible* $\{x \in S - T.\ f\ x \neq 0\}$ *negligible* $\{x \in T - S.\ f\ x \neq 0\}$
**shows** *integral* $S\ f$ = *integral* $T\ f$

### 6.14.35 More lemmas that are useful later

### 6.14.36 Continuity of the integral (for a 1-dimensional interval)

### 6.14.37 A straddling criterion for integrability

### 6.14.38 Adding integrals over several sets

### 6.14.39 Also tagged divisions

### 6.14.40 Henstock's lemma

### 6.14.41 Monotone convergence (bounded interval first)

#### 6.14.42 differentiation under the integral sign

#### 6.14.43 Exchange uniform limit and integral

#### 6.14.44 Integration by parts

#### 6.14.45 Integration by substitution

#### 6.14.46 Compute a double integral using iterated integrals and switching the order of integration

**theorem** *integral_swap_continuous*:
  **fixes** *f* :: [′*a*::*euclidean_space*, ′*b*::*euclidean_space*] ⇒ ′*c*::*banach*
  **assumes** *continuous_on* (*cbox* (*a*,*c*) (*b*,*d*)) (λ(*x*,*y*). *f x y*)
    **shows** *integral* (*cbox a b*) (λ*x*. *integral* (*cbox c d*) (*f x*)) =
        *integral* (*cbox c d*) (λ*y*. *integral* (*cbox a b*) (λ*x*. *f x y*))

#### 6.14.47 Definite integrals for exponential and power function

**end**

## 6.15 Radon-Nikodým Derivative

**theory** *Radon_Nikodym*
**imports** *Bochner_Integration*
**begin**

**definition** *diff_measure* :: ′*a measure* ⇒ ′*a measure* ⇒ ′*a measure*
**where**
  *diff_measure M N* = *measure_of* (*space M*) (*sets M*) (λ*A*. *emeasure M A* − *emeasure N A*)
**proposition** (**in** *sigma_finite_measure*) *obtain_positive_integrable_function*:
  **obtains** *f*::′*a* ⇒ *real* **where**
    *f* ∈ *borel_measurable M*
    ⋀*x*. *f x* > *0*
    ⋀*x*. *f x* ≤ *1*
    *integrable M f*

### 6.15.1 Absolutely continuous

**definition** *absolutely_continuous* :: ′*a measure* ⇒ ′*a measure* ⇒ *bool* **where**
  *absolutely_continuous M N* ⟷ *null_sets M* ⊆ *null_sets N*

### 6.15.2 Existence of the Radon-Nikodym derivative

**proposition**

(**in** *finite_measure*) *Radon_Nikodym_finite_measure*:
**assumes** *finite_measure N* **and** *sets_eq*[*simp*]: *sets N = sets M*
**assumes** *absolutely_continuous M N*
**shows** $\exists f \in$ *borel_measurable M*. *density M f = N*

**proposition** (**in** *finite_measure*) *Radon_Nikodym_finite_measure_infinite*:
**assumes** *absolutely_continuous M N* **and** *sets_eq*: *sets N = sets M*
**shows** $\exists f\in$*borel_measurable M*. *density M f = N*

**theorem** (**in** *sigma_finite_measure*) *Radon_Nikodym*:
**assumes** *ac*: *absolutely_continuous M N* **assumes** *sets_eq*: *sets N = sets M*
**shows** $\exists f \in$ *borel_measurable M*. *density M f = N*

### 6.15.3 Uniqueness of densities

**proposition** (**in** *sigma_finite_measure*) *density_unique*:
**assumes** *f*: $f \in$ *borel_measurable M*
**assumes** *f'*: $f' \in$ *borel_measurable M*
**assumes** *density_eq*: *density M f = density M f'*
**shows** *AE x in M*. $f\ x = f'\ x$

### 6.15.4 Radon-Nikodym derivative

**definition** *RN_deriv* :: *'a measure* $\Rightarrow$ *'a measure* $\Rightarrow$ *'a* $\Rightarrow$ *ennreal* **where**
*RN_deriv M N* =
(*if* $\exists f$. $f \in$ *borel_measurable M* $\wedge$ *density M f = N*
*then SOME f*. $f \in$ *borel_measurable M* $\wedge$ *density M f = N*
*else* ($\lambda$_. *0*))

**proposition** (**in** *sigma_finite_measure*) *real_RN_deriv*:
**assumes** *finite_measure N*
**assumes** *ac*: *absolutely_continuous M N sets N = sets M*
**obtains** *D* **where** $D \in$ *borel_measurable M*
**and** *AE x in M*. *RN_deriv M N x = ennreal* (*D x*)
**and** *AE x in N*. $0 < D\ x$
**and** $\bigwedge x$. $0 \le D\ x$

**end**

**theory** *Set_Integral*
**imports** *Radon_Nikodym*
**begin**

**definition** $set\_borel\_measurable\ M\ A\ f \equiv (\lambda x.\ indicator\ A\ x *_R f\ x) \in borel\_measurable\ M$

**definition** $set\_integrable\ M\ A\ f \equiv integrable\ M\ (\lambda x.\ indicator\ A\ x *_R f\ x)$

**definition** $set\_lebesgue\_integral\ M\ A\ f \equiv lebesgue\_integral\ M\ (\lambda x.\ indicator\ A\ x *_R f\ x)$

**proposition** *set_borel_measurable_subset*:
**fixes** $f :: \_ \Rightarrow \_ :: \{banach, second\_countable\_topology\}$
**assumes** [*measurable*]: $set\_borel\_measurable\ M\ A\ f\ B \in sets\ M$ **and** $B \subseteq A$
**shows** $set\_borel\_measurable\ M\ B\ f$

**proposition** *nn_integral_disjoint_family*:
**assumes** [*measurable*]: $f \in borel\_measurable\ M\ \bigwedge(n::nat).\ B\ n \in sets\ M$
**and** *disjoint_family B*
**shows** $(\int^+ x \in (\bigcup n.\ B\ n).\ f\ x\ \partial M) = (\sum n.\ (\int^+ x \in B\ n.\ f\ x\ \partial M))$
**proposition** *Scheffe_lemma1*:
**assumes** $\bigwedge n.\ integrable\ M\ (F\ n)\ integrable\ M\ f$
$AE\ x\ in\ M.\ (\lambda n.\ F\ n\ x) \longrightarrow f\ x$
$limsup\ (\lambda n.\ \int^+ x.\ norm(F\ n\ x)\ \partial M) \le (\int^+ x.\ norm(f\ x)\ \partial M)$
**shows** $(\lambda n.\ \int^+ x.\ norm(F\ n\ x - f\ x)\ \partial M) \longrightarrow 0$

**proposition** *Scheffe_lemma2*:
**fixes** $F::nat \Rightarrow 'a \Rightarrow 'b::\{banach, second\_countable\_topology\}$
**assumes** $\bigwedge n::nat.\ F\ n \in borel\_measurable\ M\ integrable\ M\ f$
$AE\ x\ in\ M.\ (\lambda n.\ F\ n\ x) \longrightarrow f\ x$
$\bigwedge n.\ (\int^+ x.\ norm(F\ n\ x)\ \partial M) \le (\int^+ x.\ norm(f\ x)\ \partial M)$
**shows** $(\lambda n.\ \int^+ x.\ norm(F\ n\ x - f\ x)\ \partial M) \longrightarrow 0$

**proposition** *tendsto_set_lebesgue_integral_at_top*:
**fixes** $f :: real \Rightarrow 'a::\{banach, second\_countable\_topology\}$
**assumes** *sets*: $\bigwedge b.\ b \ge a \Longrightarrow \{a..b\} \in sets\ M$
**and** *int*: $set\_integrable\ M\ \{a..\}\ f$
**shows** $((\lambda b.\ set\_lebesgue\_integral\ M\ \{a..b\}\ f) \longrightarrow set\_lebesgue\_integral\ M\ \{a..\}\ f)\ at\_top$

**proposition** *tendsto_set_lebesgue_integral_at_bot*:
**fixes** $f :: real \Rightarrow 'a::\{banach, second\_countable\_topology\}$
**assumes** *sets*: $\bigwedge a.\ a \le b \Longrightarrow \{a..b\} \in sets\ M$
**and** *int*: $set\_integrable\ M\ \{..b\}\ f$

**shows** $((\lambda a.\ set\_lebesgue\_integral\ M\ \{a..b\}\ f) \longrightarrow set\_lebesgue\_integral\ M\ \{..b\}\ f)\ at\_bot$

**theorem** *integral_Markov_inequality'*:
**fixes** $u :: 'a \Rightarrow real$
**assumes** [*measurable*]: $set\_integrable\ M\ A\ u$ **and** $A \in sets\ M$
**assumes** $AE\ x\ in\ M.\ x \in A \longrightarrow u\ x \geq 0$ **and** $0 < (c::real)$
**shows** $emeasure\ M\ \{x{\in}A.\ u\ x \geq c\} \leq (1/c::real) * (\int x{\in}A.\ u\ x\ \partial M)$

**theorem** *integral_Markov_inequality'_measure*:
**assumes** [*measurable*]: $set\_integrable\ M\ A\ u$ **and** $A \in sets\ M$
**and** $AE\ x\ in\ M.\ x \in A \longrightarrow 0 \leq u\ x\ 0 < (c::real)$
**shows** $measure\ M\ \{x{\in}A.\ u\ x \geq c\} \leq (\int x{\in}A.\ u\ x\ \partial M)\ /\ c$

**theorem** (**in** *finite_measure*) *Chernoff_ineq_ge*:
**assumes** $s$: $s > 0$
**assumes** *integrable*: $set\_integrable\ M\ A\ (\lambda x.\ exp\ (s * f\ x))$ **and** $A \in sets\ M$
**shows** $measure\ M\ \{x{\in}A.\ f\ x \geq a\} \leq exp\ (-s * a) * (\int x{\in}A.\ exp\ (s * f\ x)\ \partial M)$
**proof** −
**have** $\{x{\in}A.\ f\ x \geq a\} = \{x{\in}A.\ exp\ (s * f\ x) \geq exp\ (s * a)\}$
**using** $s$ **by** *auto*
**also have** $measure\ M\ \ldots \leq set\_lebesgue\_integral\ M\ A\ (\lambda x.\ exp\ (s * f\ x))\ /\ exp\ (s * a)$
**by** (*intro integral_Markov_inequality'_measure assms*) *auto*
**finally show** *?thesis*
**by** (*simp add*: *exp_minus field_simps*)
**qed**

**theorem** (**in** *finite_measure*) *Chernoff_ineq_le*:
**assumes** $s$: $s > 0$
**assumes** *integrable*: $set\_integrable\ M\ A\ (\lambda x.\ exp\ (-s * f\ x))$ **and** $A \in sets\ M$
**shows** $measure\ M\ \{x{\in}A.\ f\ x \leq a\} \leq exp\ (s * a) * (\int x{\in}A.\ exp\ (-s * f\ x)\ \partial M)$
**proof** −
**have** $\{x{\in}A.\ f\ x \leq a\} = \{x{\in}A.\ exp\ (-s * f\ x) \geq exp\ (-s * a)\}$
**using** $s$ **by** *auto*
**also have** $measure\ M\ \ldots \leq set\_lebesgue\_integral\ M\ A\ (\lambda x.\ exp\ (-s * f\ x))\ /\ exp\ (-s * a)$
**by** (*intro integral_Markov_inequality'_measure assms*) *auto*
**finally show** *?thesis*
**by** (*simp add*: *exp_minus field_simps*)
**qed**

**end**

## 6.16 Homeomorphism Theorems

**theory** *Homeomorphism*

**imports** *Homotopy*
**begin**

### 6.16.1 Homeomorphism of all convex compact sets with nonempty interior

**proposition**
**fixes** $S$ :: *'a::euclidean_space set*
**assumes** *compact S* **and** *0*: $0 \in$ *rel_interior S*
**and** *star*: $\bigwedge x.\ x \in S \Longrightarrow$ *open_segment 0 x* $\subseteq$ *rel_interior S*
**shows** *starlike_compact_projective1_0*:
$S -$ *rel_interior S homeomorphic sphere 0 1* $\cap$ *affine hull S*
(**is** *?SMINUS homeomorphic ?SPHER*)
**and** *starlike_compact_projective2_0*:
*S homeomorphic cball 0 1* $\cap$ *affine hull S*
(**is** *S homeomorphic ?CBALL*)

**corollary**
**fixes** $S$ :: *'a::euclidean_space set*
**assumes** *compact S* **and** *a*: $a \in$ *rel_interior S*
**and** *star*: $\bigwedge x.\ x \in S \Longrightarrow$ *open_segment a x* $\subseteq$ *rel_interior S*
**shows** *starlike_compact_projective1*:
$S -$ *rel_interior S homeomorphic sphere a 1* $\cap$ *affine hull S*
**and** *starlike_compact_projective2*:
*S homeomorphic cball a 1* $\cap$ *affine hull S*

**corollary** *starlike_compact_projective_special*:
**assumes** *compact S*
**and** *cb01*: *cball (0::'a::euclidean_space) 1* $\subseteq S$
**and** *scale*: $\bigwedge x\ u.\ [\![x \in S;\ 0 \le u;\ u < 1]\!] \Longrightarrow u *_R x \in S -$ *frontier S*
**shows** *S homeomorphic (cball (0::'a::euclidean_space) 1)*

### 6.16.2 Homeomorphisms between punctured spheres and affine sets

**theorem** *homeomorphic_punctured_affine_sphere_affine*:
**fixes** $a$ :: *'a :: euclidean_space*
**assumes** $0 < r$ $b \in$ *sphere a r affine T* $a \in T$ $b \in T$ *affine p*
**and** *aff*: *aff_dim T = aff_dim p + 1*
**shows** *(sphere a r* $\cap$ *T)* $- \{b\}$ *homeomorphic p*

**corollary** *homeomorphic_punctured_sphere_affine*:
**fixes** $a$ :: *'a :: euclidean_space*
**assumes** $0 < r$ **and** *b*: $b \in$ *sphere a r*
**and** *affine T* **and** *affS*: *aff_dim T + 1 = DIM('a)*
**shows** *(sphere a r* $- \{b\}$*) homeomorphic T*

**corollary** *homeomorphic_punctured_sphere_hyperplane*:
**fixes** *a* :: *'a* :: *euclidean_space*
**assumes** *0* < *r* **and** *b*: *b* ∈ *sphere a r*
**and** *c* ≠ *0*
**shows** (*sphere a r* − {*b*}) *homeomorphic* {*x*::*'a*. *c* · *x* = *d*}

**proposition** *homeomorphic_punctured_sphere_affine_gen*:
**fixes** *a* :: *'a* :: *euclidean_space*
**assumes** *convex S bounded S* **and** *a*: *a* ∈ *rel_frontier S*
**and** *affine T* **and** *affS*: *aff_dim S* = *aff_dim T* + *1*
**shows** *rel_frontier S* − {*a*} *homeomorphic T*
**proposition** *homeomorphic_closedin_convex*:
**fixes** *S* :: *'m*::*euclidean_space set*
**assumes** *aff_dim S* < *DIM*(*'n*)
**obtains** *U* **and** *T* :: *'n*::*euclidean_space set*
**where** *convex U U* ≠ {} *closedin* (*top_of_set U*) *T*
*S homeomorphic T*

### 6.16.3 Locally compact sets in an open set

**proposition** *locally_compact_homeomorphic_closed*:
**fixes** *S* :: *'a*::*euclidean_space set*
**assumes** *locally compact S* **and** *dimlt*: *DIM*(*'a*) < *DIM*(*'b*)
**obtains** *T* :: *'b*::*euclidean_space set* **where** *closed T S homeomorphic T*

**proposition** *homeomorphic_convex_compact_cball*:
**fixes** *e* :: *real*
**and** *S* :: *'a*::*euclidean_space set*
**assumes** *S*: *convex S compact S interior S* ≠ {} **and** *e* > *0*
**shows** *S homeomorphic* (*cball* (*b*::*'a*) *e*)

**corollary** *homeomorphic_convex_compact*:
**fixes** *S* :: *'a*::*euclidean_space set*
**and** *T* :: *'a set*
**assumes** *convex S compact S interior S* ≠ {}
**and** *convex T compact T interior T* ≠ {}
**shows** *S homeomorphic T*

### 6.16.4 Covering spaces and lifting results for them

**definition** *covering_space*
:: *'a*::*topological_space set* ⇒ (*'a* ⇒ *'b*) ⇒ *'b*::*topological_space set* ⇒ *bool*
**where**

*covering_space c p S ≡*
*continuous_on c p ∧ p ‘ c = S ∧*
*(∀ x ∈ S. ∃ T. x ∈ T ∧ openin (top_of_set S) T ∧*
*(∃ v. ⋃ v = c ∩ p −‘ T ∧*
*(∀ u ∈ v. openin (top_of_set c) u) ∧*
*pairwise disjnt v ∧*
*(∀ u ∈ v. ∃ q. homeomorphism u T p q)))*

**proposition** *covering_space_open_map*:
**fixes** *S* :: *'a* :: *metric_space set* **and** *T* :: *'b* :: *metric_space set*
**assumes** *p*: *covering_space c p S* **and** *T*: *openin (top_of_set c) T*
**shows** *openin (top_of_set S) (p ‘ T)*

**proposition** *covering_space_lift_unique*:
**fixes** *f* :: *'a::topological_space ⇒ 'b::topological_space*
**fixes** *g1* :: *'a ⇒ 'c::real_normed_vector*
**assumes** *covering_space c p S*
*g1 a = g2 a*
*continuous_on T f  f ∈ T → S*
*continuous_on T g1  g1 ∈ T → c  ⋀x. x ∈ T ⟹ f x = p(g1 x)*
*continuous_on T g2  g2 ∈ T → c  ⋀x. x ∈ T ⟹ f x = p(g2 x)*
*connected T  a ∈ T  x ∈ T*
**shows** *g1 x = g2 x*

**proposition** *covering_space_locally_eq*:
**fixes** *p* :: *'a::real_normed_vector ⇒ 'b::real_normed_vector*
**assumes** *cov*: *covering_space C p S*
**and** *pim*: *⋀T. ⟦T ⊆ C; φ T⟧ ⟹ ψ(p ‘ T)*
**and** *qim*: *⋀q U. ⟦U ⊆ S; continuous_on U q; ψ U⟧ ⟹ φ(q ‘ U)*
**shows** *locally ψ S ⟷ locally φ C*
(**is** *?lhs = ?rhs*)

**proposition** *covering_space_lift_homotopy*:
**fixes** *p* :: *'a::real_normed_vector ⇒ 'b::real_normed_vector*
**and** *h* :: *real × 'c::real_normed_vector ⇒ 'b*
**assumes** *cov*: *covering_space C p S*
**and** *conth*: *continuous_on ({0..1} × U) h*
**and** *him*: *h ∈ ({0..1} × U) → S*
**and** *heq*: *⋀y. y ∈ U ⟹ h (0,y) = p(f y)*
**and** *contf*: *continuous_on U f* **and** *fim*: *f ∈ U → C*
**obtains** *k* **where** *continuous_on ({0..1} × U) k*
*k ∈ ({0..1} × U) → C*
*⋀y. y ∈ U ⟹ k(0, y) = f y*
*⋀z. z ∈ {0..1} × U ⟹ h z = p(k z)*

**corollary** *covering_space_lift_homotopy_alt*:
**fixes** $p$ :: $'a$::*real_normed_vector* ⇒ $'b$::*real_normed_vector*
**and** $h$ :: $'c$::*real_normed_vector* × *real* ⇒ $'b$
**assumes** *cov*: *covering_space* $C$ $p$ $S$
**and** *conth*: *continuous_on* $(U \times \{0..1\})$ $h$
**and** *him*: $h \in (U \times \{0..1\}) \to S$
**and** *heq*: $\bigwedge y.\ y \in U \Longrightarrow h\ (y,0) = p(f\ y)$
**and** *contf*: *continuous_on* $U$ $f$ **and** *fim*: $f \in U \to C$
**obtains** $k$ **where** *continuous_on* $(U \times \{0..1\})$ $k$
$k \in (U \times \{0..1\}) \to C$
$\bigwedge y.\ y \in U \Longrightarrow k(y,\ 0) = f\ y$
$\bigwedge z.\ z \in U \times \{0..1\} \Longrightarrow h\ z = p(k\ z)$

**corollary** *covering_space_lift_homotopic_function*:
**fixes** $p$ :: $'a$::*real_normed_vector* ⇒ $'b$::*real_normed_vector* **and** $g$:: $'c$::*real_normed_vector* ⇒ $'a$
**assumes** *cov*: *covering_space* $C$ $p$ $S$
**and** *contg*: *continuous_on* $U$ $g$
**and** *gim*: $g \in U \to C$
**and** *pgeq*: $\bigwedge y.\ y \in U \Longrightarrow p(g\ y) = f\ y$
**and** *hom*: *homotopic_with_canon* (λ*x*. *True*) $U$ $S$ $f$ $f'$
**obtains** $g'$ **where** *continuous_on* $U$ $g'$ *image* $g'$ $U \subseteq C$ $\bigwedge y.\ y \in U \Longrightarrow p(g'\ y) = f'\ y$

**corollary** *covering_space_lift_inessential_function*:
**fixes** $p$ :: $'a$::*real_normed_vector* ⇒ $'b$::*real_normed_vector* **and** $U$ :: $'c$::*real_normed_vector* *set*
**assumes** *cov*: *covering_space* $C$ $p$ $S$
**and** *hom*: *homotopic_with_canon* (λ*x*. *True*) $U$ $S$ $f$ (λ*x*. *a*)
**obtains** $g$ **where** *continuous_on* $U$ $g$ $g$ ‘ $U \subseteq C$ $\bigwedge y.\ y \in U \Longrightarrow p(g\ y) = f\ y$

### 6.16.5 Lifting of general functions to covering space

**proposition** *covering_space_lift_path_strong*:
**fixes** $p$ :: $'a$::*real_normed_vector* ⇒ $'b$::*real_normed_vector*
**and** $f$ :: $'c$::*real_normed_vector* ⇒ $'b$
**assumes** *cov*: *covering_space* $C$ $p$ $S$ **and** $a \in C$
**and** *path* $g$ **and** *pag*: *path_image* $g \subseteq S$ **and** *pas*: *pathstart* $g = p\ a$
**obtains** $h$ **where** *path* $h$ *path_image* $h \subseteq C$ *pathstart* $h = a$
**and** $\bigwedge t.\ t \in \{0..1\} \Longrightarrow p(h\ t) = g\ t$

**corollary** *covering_space_lift_path*:
**fixes** $p$ :: $'a$::*real_normed_vector* ⇒ $'b$::*real_normed_vector*
**assumes** *cov*: *covering_space* $C$ $p$ $S$ **and** *path* $g$ **and** *pig*: *path_image* $g \subseteq S$
**obtains** $h$ **where** *path* $h$ *path_image* $h \subseteq C$ $\bigwedge t.\ t \in \{0..1\} \Longrightarrow p(h\ t) = g\ t$

**proposition** *covering_space_lift_homotopic_paths*:

**fixes** *p* :: *'a::real_normed_vector* ⇒ *'b::real_normed_vector*
**assumes** *cov*: *covering_space C p S*
**and** *path g1* **and** *pig1*: *path_image g1* ⊆ *S*
**and** *path g2* **and** *pig2*: *path_image g2* ⊆ *S*
**and** *hom*: *homotopic_paths S g1 g2*
**and** *path h1* **and** *pih1*: *path_image h1* ⊆ *C* **and** *ph1*: ⋀*t*. *t* ∈ {*0..1*} ⟹ *p*(*h1 t*) = *g1 t*
**and** *path h2* **and** *pih2*: *path_image h2* ⊆ *C* **and** *ph2*: ⋀*t*. *t* ∈ {*0..1*} ⟹ *p*(*h2 t*) = *g2 t*
**and** *h1h2*: *pathstart h1* = *pathstart h2*
**shows** *homotopic_paths C h1 h2*

**corollary** *covering_space_monodromy*:
**fixes** *p* :: *'a::real_normed_vector* ⇒ *'b::real_normed_vector*
**assumes** *cov*: *covering_space C p S*
**and** *path g1* **and** *pig1*: *path_image g1* ⊆ *S*
**and** *path g2* **and** *pig2*: *path_image g2* ⊆ *S*
**and** *hom*: *homotopic_paths S g1 g2*
**and** *path h1* **and** *pih1*: *path_image h1* ⊆ *C* **and** *ph1*: ⋀*t*. *t* ∈ {*0..1*} ⟹ *p*(*h1 t*) = *g1 t*
**and** *path h2* **and** *pih2*: *path_image h2* ⊆ *C* **and** *ph2*: ⋀*t*. *t* ∈ {*0..1*} ⟹ *p*(*h2 t*) = *g2 t*
**and** *h1h2*: *pathstart h1* = *pathstart h2*
**shows** *pathfinish h1* = *pathfinish h2*

**corollary** *covering_space_lift_homotopic_path*:
**fixes** *p* :: *'a::real_normed_vector* ⇒ *'b::real_normed_vector*
**assumes** *cov*: *covering_space C p S*
**and** *hom*: *homotopic_paths S f f'*
**and** *path g* **and** *pig*: *path_image g* ⊆ *C*
**and** *a*: *pathstart g* = *a* **and** *b*: *pathfinish g* = *b*
**and** *pgeq*: ⋀*t*. *t* ∈ {*0..1*} ⟹ *p*(*g t*) = *f t*
**obtains** *g'* **where** *path g' path_image g'* ⊆ *C*
*pathstart g'* = *a pathfinish g'* = *b* ⋀*t*. *t* ∈ {*0..1*} ⟹ *p*(*g' t*) = *f' t*

**proposition** *covering_space_lift_general*:
**fixes** *p* :: *'a::real_normed_vector* ⇒ *'b::real_normed_vector*
**and** *f* :: *'c::real_normed_vector* ⇒ *'b*
**assumes** *cov*: *covering_space C p S* **and** *a* ∈ *C z* ∈ *U*
**and** *U*: *path_connected U locally path_connected U*
**and** *contf*: *continuous_on U f* **and** *fim*: *f* ∈ *U* → *S*
**and** *feq*: *f z* = *p a*
**and** *hom*: ⋀*r*. ⟦*path r*; *path_image r* ⊆ *U*; *pathstart r* = *z*; *pathfinish r* = *z*⟧
⟹ ∃ *q*. *path q* ∧ *path_image q* ⊆ *C* ∧
*pathstart q* = *a* ∧ *pathfinish q* = *a* ∧
*homotopic_paths S* (*f* ∘ *r*) (*p* ∘ *q*)

**obtains** *g* **where** *continuous_on U g g* ∈ *U* → *C g z* = *a* ⋀*y*. *y* ∈ *U* ⟹ *p*(*g y*) = *f y*

**corollary** *covering_space_lift_stronger*:
**fixes** *p* :: *'a::real_normed_vector* ⇒ *'b::real_normed_vector*
**and** *f* :: *'c::real_normed_vector* ⇒ *'b*
**assumes** *cov*: *covering_space C p S a* ∈ *C z* ∈ *U*
**and** *U*: *path_connected U locally path_connected U*
**and** *contf*: *continuous_on U f* **and** *fim*: *f* ∈ *U* → *S*
**and** *feq*: *f z* = *p a*
**and** *hom*: ⋀*r*. ⟦*path r*; *path_image r* ⊆ *U*; *pathstart r* = *z*; *pathfinish r* = *z*⟧ ⟹ ∃ *b*. *homotopic_paths S* (*f* ∘ *r*) (*linepath b b*)
**obtains** *g* **where** *continuous_on U g g* ∈ *U* → *C g z* = *a* ⋀*y*. *y* ∈ *U* ⟹ *p*(*g y*) = *f y*

**corollary** *covering_space_lift_strong*:
**fixes** *p* :: *'a::real_normed_vector* ⇒ *'b::real_normed_vector*
**and** *f* :: *'c::real_normed_vector* ⇒ *'b*
**assumes** *cov*: *covering_space C p S a* ∈ *C z* ∈ *U*
**and** *scU*: *simply_connected U* **and** *lpcU*: *locally path_connected U*
**and** *contf*: *continuous_on U f* **and** *fim*: *f* ∈ *U* → *S*
**and** *feq*: *f z* = *p a*
**obtains** *g* **where** *continuous_on U g g* ∈ *U* → *C g z* = *a* ⋀*y*. *y* ∈ *U* ⟹ *p*(*g y*) = *f y*

**corollary** *covering_space_lift*:
**fixes** *p* :: *'a::real_normed_vector* ⇒ *'b::real_normed_vector*
**and** *f* :: *'c::real_normed_vector* ⇒ *'b*
**assumes** *cov*: *covering_space C p S*
**and** *U*: *simply_connected U locally path_connected U*
**and** *contf*: *continuous_on U f* **and** *fim*: *f* ∈ *U* → *S*
**obtains** *g* **where** *continuous_on U g g* ∈ *U* → *C* ⋀*y*. *y* ∈ *U* ⟹ *p*(*g y*) = *f y*

**end**

**theory** *Equivalence_Lebesgue_Henstock_Integration*
**imports**
*Lebesgue_Measure*
*Henstock_Kurzweil_Integration*
*Complete_Measure*
*Set_Integral*
*Homeomorphism*
*Cartesian_Euclidean_Space*
**begin**

### 6.16.6 Equivalence Lebesgue integral on *lborel* and HK-integral

### 6.16.7 Absolute integrability (this is the same as Lebesgue integrability)

### 6.16.8 Applications to Negligibility

**corollary** *eventually_ae_filter_negligible*:
  *eventually P (ae_filter lebesgue)* $\longleftrightarrow$ $(\exists\, N.$ *negligible N* $\wedge$ $\{x.\ \neg\ P\ x\} \subseteq N)$

**proposition** *negligible_convex_frontier*:
  **fixes** $S$ :: $'N$ :: *euclidean_space set*
  **assumes** *convex S*
    **shows** *negligible*(*frontier S*)

**corollary** *negligible_sphere*: *negligible* (*sphere a e*)

**proposition** *open_not_negligible*:
  **assumes** *open S* $S \neq \{\}$
  **shows** $\neg$ *negligible S*

### 6.16.9 Negligibility of image under non-injective linear map

### 6.16.10 Negligibility of a Lipschitz image of a negligible set

**proposition** *negligible_locally_Lipschitz_image*:
  **fixes** $f$ :: $'M$::*euclidean_space* $\Rightarrow$ $'N$::*euclidean_space*
  **assumes** *MleN*: *DIM*$('M) \leq$ *DIM*$('N)$ *negligible S*
    **and** *lips*: $\bigwedge x.\ x \in S$
        $\Longrightarrow \exists\, T\ B.$ *open* $T \wedge x \in T \wedge$
          $(\forall\, y \in S \cap T.$ *norm*$(f\ y - f\ x) \leq B *$ *norm*$(y - x))$
    **shows** *negligible* $(f\ `\ S)$

**corollary** *negligible_differentiable_image_negligible*:
  **fixes** $f$ :: $'M$::*euclidean_space* $\Rightarrow$ $'N$::*euclidean_space*
  **assumes** *MleN*: *DIM*$('M) \leq$ *DIM*$('N)$ *negligible S*
    **and** *diff_f*: *f differentiable_on S*
    **shows** *negligible* $(f\ `\ S)$

**corollary** *negligible_differentiable_image_lowdim*:
  **fixes** $f$ :: $'M$::*euclidean_space* $\Rightarrow$ $'N$::*euclidean_space*
  **assumes** *MlessN*: *DIM*$('M) <$ *DIM*$('N)$ **and** *diff_f*: *f differentiable_on S*
    **shows** *negligible* $(f\ `\ S)$

### 6.16.11 Measurability of countable unions and intersections of various kinds.

### 6.16.12 Negligibility is a local property

### 6.16.13 Integral bounds

**proposition** *bounded_variation_absolutely_integrable_interval*:
**fixes** $f$ :: $'n::euclidean\_space \Rightarrow 'm::euclidean\_space$
**assumes** $f$: $f$ *integrable_on cbox a b*
**and** $*$: $\bigwedge d.\ d\ division\_of\ (cbox\ a\ b) \Longrightarrow sum\ (\lambda K.\ norm(integral\ K\ f))\ d \leq B$
**shows** $f$ *absolutely_integrable_on cbox a b*

### 6.16.14 Outer and inner approximation of measurable sets by well-behaved sets.

**proposition** *measurable_outer_intervals_bounded*:
**assumes** $S \in lmeasurable\ S \subseteq cbox\ a\ b\ e > 0$
**obtains** $\mathcal{D}$
**where** *countable* $\mathcal{D}$
$\bigwedge K.\ K \in \mathcal{D} \Longrightarrow K \subseteq cbox\ a\ b \wedge K \neq \{\} \wedge (\exists c\ d.\ K = cbox\ c\ d)$
$pairwise\ (\lambda A\ B.\ interior\ A \cap interior\ B = \{\})\ \mathcal{D}$
$\bigwedge u\ v.\ cbox\ u\ v \in \mathcal{D} \Longrightarrow \exists n.\ \forall i \in Basis.\ v \cdot i - u \cdot i = (b \cdot i - a \cdot i)/2\hat{}n$
$\bigwedge K.\ [\![K \in \mathcal{D};\ box\ a\ b \neq \{\}]\!] \Longrightarrow interior\ K \neq \{\}$
$S \subseteq \bigcup\mathcal{D}\ \bigcup\mathcal{D} \in lmeasurable\ measure\ lebesgue\ (\bigcup\mathcal{D}) \leq measure\ lebesgue\ S$
$+\ e$

### 6.16.15 Transformation of measure by linear maps

**proposition** *measure_linear_sufficient*:
**fixes** $f$ :: $'n::euclidean\_space \Rightarrow 'n$
**assumes** *linear f* **and** $S$: $S \in lmeasurable$
**and** *im*: $\bigwedge a\ b.\ measure\ lebesgue\ (f\ `\ (cbox\ a\ b)) = m * measure\ lebesgue\ (cbox$
$a\ b)$
**shows** $f\ `\ S \in lmeasurable \wedge m * measure\ lebesgue\ S = measure\ lebesgue\ (f\ `\ S)$

### 6.16.16 Lemmas about absolute integrability

**corollary** *absolutely_integrable_on_const* [*simp*]:
**fixes** $c$ :: $'a::euclidean\_space$
**assumes** $S \in lmeasurable$
**shows** $(\lambda x.\ c)$ *absolutely_integrable_on S*

### 6.16.17 Componentwise

**proposition** *absolutely_integrable_componentwise_iff*:
**shows** *f absolutely_integrable_on A* $\longleftrightarrow$ ($\forall$ *b*$\in$*Basis*. ($\lambda x$. *f x* $\cdot$ *b*) *absolutely_integrable_on A*)

**corollary** *absolutely_integrable_max_1*:
**fixes** *f* :: *'n::euclidean_space* $\Rightarrow$ *real*
**assumes** *f absolutely_integrable_on S g absolutely_integrable_on S*
**shows** ($\lambda x$. *max* (*f x*) (*g x*)) *absolutely_integrable_on S*

**corollary** *absolutely_integrable_min_1*:
**fixes** *f* :: *'n::euclidean_space* $\Rightarrow$ *real*
**assumes** *f absolutely_integrable_on S g absolutely_integrable_on S*
**shows** ($\lambda x$. *min* (*f x*) (*g x*)) *absolutely_integrable_on S*

### 6.16.18 Dominated convergence

**proposition** *integral_countable_UN*:
**fixes** *f* :: *real^'m* $\Rightarrow$ *real^'n*
**assumes** *f*: *f absolutely_integrable_on* ($\bigcup$(*range s*))
**and** *s*: $\bigwedge$*m*. *s m* $\in$ *sets lebesgue*
**shows** $\bigwedge$*n*. *f absolutely_integrable_on* ($\bigcup$ *m*$\le$*n*. *s m*)
**and** ($\lambda n$. *integral* ($\bigcup$ *m*$\le$*n*. *s m*) *f*) $\longrightarrow$ *integral* ($\bigcup$(*s* ` *UNIV*)) *f* (**is** *?F* $\longrightarrow$ *?I*)

### 6.16.19 Fundamental Theorem of Calculus for the Lebesgue integral

### 6.16.20 Integration by parts

### 6.16.21 A non-negative continuous function whose integral is zero must be zero

**corollary** *integral_cbox_eq_0_iff*:
**fixes** *f* :: *'a::euclidean_space* $\Rightarrow$ *real*
**assumes** *continuous_on* (*cbox a b*) *f* **and** *box a b* $\neq$ {}
**and** $\bigwedge$*x*. *x* $\in$ *cbox a b* $\Longrightarrow$ *f x* $\ge$ *0*
**shows** *integral* (*cbox a b*) *f* = *0* $\longleftrightarrow$ ($\forall$ *x* $\in$ *cbox a b*. *f x* = *0*) (**is** *?lhs* = *?rhs*)

### 6.16.22 Various common equivalent forms of function measurability

### 6.16.23 Lebesgue sets and continuous images

**proposition** *lebesgue_regular_inner*:
**assumes** $S \in$ *sets lebesgue*
**obtains** $K$ $C$ **where** *negligible* $K$ $\bigwedge n{::}nat.$ *compact*$(C\ n)$ $S = (\bigcup n.\ C\ n) \cup K$

### 6.16.24 Affine lemmas

**lemma** *lebesgue_integral_real_affine*:
**fixes** $f$ :: *real* $\Rightarrow$ $'a$ :: *euclidean_space* **and** $c$ :: *real*
**assumes** $c$: $c \neq 0$ **shows** $(\int x.\ f\ x\ \partial\ lebesgue) = |c| *_R (\int x.\ f(t + c * x)\ \partial lebesgue)$

### 6.16.25 More results on integrability

**proposition** *measurable_bounded_by_integrable_imp_integrable*:
**fixes** $f$ :: $'a$::*euclidean_space* $\Rightarrow$ $'b$::*euclidean_space*
**assumes** $f$: $f \in$ *borel_measurable* (*lebesgue_on* $S$) **and** $g$: $g$ *integrable_on* $S$
**and** *normf*: $\bigwedge x.\ x \in S \Longrightarrow norm(f\ x) \leq g\ x$ **and** $S$: $S \in$ *sets lebesgue*
**shows** $f$ *integrable_on* $S$

### 6.16.26 Relation between Borel measurability and integrability.

**proposition** *negligible_differentiable_vimage*:
**fixes** $f$ :: $'a \Rightarrow 'a$::*euclidean_space*
**assumes** *negligible* $T$
**and** $f'$: $\bigwedge x.\ x \in S \Longrightarrow inj(f'\ x)$
**and** *derf*: $\bigwedge x.\ x \in S \Longrightarrow (f$ *has_derivative* $f'\ x)$ (*at* $x$ *within* $S$)
**shows** *negligible* $\{x \in S.\ f\ x \in T\}$
**proposition** *has_derivative_inverse_within*:
**fixes** $f$ :: $'a$::*real_normed_vector* $\Rightarrow$ $'b$::*euclidean_space*
**assumes** *der_f*: ($f$ *has_derivative* $f'$) (*at* $a$ *within* $S$)
**and** *cont_g*: *continuous* (*at* $(f\ a)$ *within* $f$ ‘ $S$) $g$
**and** $a \in S$ *linear* $g'$ **and** *id*: $g' \circ f' = id$
**and** *gf*: $\bigwedge x.\ x \in S \Longrightarrow g(f\ x) = x$
**shows** ($g$ *has_derivative* $g'$) (*at* $(f\ a)$ *within* $f$ ‘ $S$)

**end**

## 6.17 Complex Analysis Basics

**theory** *Complex_Analysis_Basics*
**imports** *Derivative HOL−Library.Nonpos_Ints Uncountable_Sets*
**begin**

### 6.17.1 Holomorphic functions

**definition** *holomorphic_on* :: [*complex* ⇒ *complex*, *complex set*] ⇒ *bool*
(**infixl** (*holomorphic′_on*) *50*)
**where** *f holomorphic_on s* ≡ ∀ *x*∈*s*. *f field_differentiable* (*at x within s*)

**named_theorems** *holomorphic_intros structural introduction rules for holomorphic_on*

### 6.17.2 Analyticity on a set

**definition** *analytic_on* (**infixl** (*analytic′_on*) *50*)
**where** *f analytic_on S* ≡ ∀ *x* ∈ *S*. ∃ *e*. *0* < *e* ∧ *f holomorphic_on* (*ball x e*)

**named_theorems** *analytic_intros introduction rules for proving analyticity*

**end**

## 6.18 Complex Transcendental Functions

**theory** *Complex_Transcendental*
**imports**
*Complex_Analysis_Basics Summation_Tests HOL−Library.Periodic_Fun*
**begin**

### 6.18.1 Möbius transformations

**definition** *moebius a b c d* ≡ (λ*z*. (*a*∗*z*+*b*) / (*c*∗*z*+*d* :: ′*a* :: *field*))

**theorem** *moebius_inverse*:
**assumes** *a* ∗ *d* ≠ *b* ∗ *c* *c* ∗ *z* + *d* ≠ *0*
**shows** *moebius d* (−*b*) (−*c*) *a* (*moebius a b c d z*) = *z*

### 6.18.2 Euler and de Moivre formulas

**theorem** *exp_Euler*: *exp*(i ∗ *z*) = *cos*(*z*) + i ∗ *sin*(*z*)

**theorem** *Euler*: $exp(z) = of\_real(exp(Re\ z)) *$
$(of\_real(cos(Im\ z)) + i * of\_real(sin(Im\ z)))$

### 6.18.3 The argument of a complex number (HOL Light version)

**definition** *is_Arg* :: [*complex*,*real*] ⇒ *bool*
**where** $is\_Arg\ z\ r \equiv z = of\_real(norm\ z) * exp(i * of\_real\ r)$

**definition** *Arg2pi* :: *complex* ⇒ *real*
**where** $Arg2pi\ z \equiv if\ z = 0\ then\ 0\ else\ THE\ t.\ 0 \le t \wedge t < 2*pi \wedge is\_Arg\ z\ t$

### 6.18.4 The principal branch of the Complex logarithm

**instantiation** *complex* :: *ln*
**begin**

**definition** *ln_complex* :: *complex* ⇒ *complex*
**where** $ln\_complex \equiv \lambda z.\ THE\ w.\ exp\ w = z\ \&\ -pi < Im(w)\ \&\ Im(w) \le pi$
**theorem** *Ln_series*:
**fixes** *z* :: *complex*
**assumes** $norm\ z < 1$
**shows** $(\lambda n.\ (-1)\hat{}Suc\ n\ /\ of\_nat\ n * z\hat{}n)\ sums\ ln\ (1 + z)$ (**is** $(\lambda n.\ ?f\ n * z\hat{}n)\ sums\ \_$)

**corollary** *norm_Ln_prod_le*:
**fixes** *f* :: 'a ⇒ *complex*
**assumes** $\bigwedge x.\ x \in A \Longrightarrow f\ x \neq 0$
**shows** $cmod\ (Ln\ (prod\ f\ A)) \le (\sum x \in A.\ cmod\ (Ln\ (f\ x)))$

### 6.18.5 The Argument of a Complex Number

**lemma** *Arg_def*:
**shows** $Arg\ z = (if\ z = 0\ then\ 0\ else\ Im\ (Ln\ z))$

### 6.18.6 The Unwinding Number and the Ln product Formula

**definition** *unwinding* :: *complex* ⇒ *int* **where**
$unwinding\ z \equiv THE\ k.\ of\_int\ k = (z - Ln(exp\ z))\ /\ (of\_real(2*pi) * i)$

### 6.18.7 Characterisation of *Im* (*Ln z*) (Wenda Li)

### 6.18.8 Complex arctangent

**definition** *Arctan* :: *complex* ⇒ *complex* **where**
*Arctan* ≡ λ*z*. (i/*2*) ∗ *Ln*((*1* − i∗*z*) / (*1* + i∗*z*))

**theorem** *Arctan_series*:
**assumes** *z*: *norm* (*z* :: *complex*) < *1*
**defines** *g* ≡ λ*n*. *if odd n then* −i∗i^*n* / *n else 0*
**defines** *h* ≡ λ*z n*. (−*1*)^*n* / *of_nat* (*2*∗*n*+*1*) ∗ (*z*::*complex*)^(*2*∗*n*+*1*)
**shows** (λ*n*. *g n* ∗ *z*^*n*) *sums Arctan z*
**and** *h z sums Arctan z*
**theorem** *ln_series_quadratic*:
**assumes** *x*: *x* > (*0*::*real*)
**shows** (λ*n*. (*2*∗((*x* − *1*) / (*x* + *1*)) ^ (*2*∗*n*+*1*) / *of_nat* (*2*∗*n*+*1*))) *sums ln x*

### 6.18.9 Inverse Sine

**definition** *Arcsin* :: *complex* ⇒ *complex* **where**
*Arcsin* ≡ λ*z*. −i ∗ *Ln*(i ∗ *z* + *csqrt*(*1* − $z^2$))

### 6.18.10 Inverse Cosine

**definition** *Arccos* :: *complex* ⇒ *complex* **where**
*Arccos* ≡ λ*z*. −i ∗ *Ln*(*z* + i ∗ *csqrt*(*1* − $z^2$))

### 6.18.11 Roots of unity

**theorem** *complex_root_unity*:
**fixes** *j*::*nat*
**assumes** *n* ≠ *0*
**shows** *exp*(*2* ∗ *of_real pi* ∗ i ∗ *of_nat j* / *of_nat n*)^*n* = *1*

**corollary** *bij_betw_roots_unity*:
*bij_betw* (λ*j*. *exp*(*2* ∗ *of_real pi* ∗ i ∗ *of_nat j* / *of_nat n*))
{..<*n*} {*exp*(*2* ∗ *of_real pi* ∗ i ∗ *of_nat j* / *of_nat n*) | *j*. *j* < *n*}

**end**

## 6.19 Harmonic Numbers

**theory** *Harmonic_Numbers*
**imports**
*Complex_Transcendental*
*Summation_Tests*
**begin**

### 6.19.1 The Harmonic numbers

**definition** *harm* :: *nat* ⇒ *'a* :: *real_normed_field* **where**
*harm n* = (∑ *k*=*1*..*n*. *inverse* (*of_nat k*))

**theorem** *not_convergent_harm*: ¬*convergent* (*harm* :: *nat* ⇒ *'a* :: *real_normed_field*)

### 6.19.2 The Euler-Mascheroni constant

**lemma** *euler_mascheroni_LIMSEQ*:
(λ*n*. *harm n* − *ln* (*of_nat n*) :: *real*) ⟶ *euler_mascheroni*

**theorem** *alternating_harmonic_series_sums*: (λ*k*. (−*1*)^*k* / *real_of_nat* (*Suc k*)) *sums ln 2*

**end**

## 6.20 The Gamma Function

**theory** *Gamma_Function*
**imports**
*Equivalence_Lebesgue_Henstock_Integration*
*Summation_Tests*
*Harmonic_Numbers*
*HOL−Library.Nonpos_Ints*
*HOL−Library.Periodic_Fun*
**begin**

### 6.20.1 The Euler form and the logarithmic Gamma function

**definition** *Gamma_series* :: (*'a* :: {*banach*,*real_normed_field*}) ⇒ *nat* ⇒ *'a* **where**
*Gamma_series z n* = *fact n* ∗ *exp* (*z* ∗ *of_real* (*ln* (*of_nat n*))) / *pochhammer z* (*n*+*1*)
**definition** *ln_Gamma_series* :: (*'a* :: {*banach*,*real_normed_field*,*ln*}) ⇒ *nat* ⇒ *'a* **where**
*ln_Gamma_series z n* = *z* ∗ *ln* (*of_nat n*) − *ln z* − (∑ *k*=*1*..*n*. *ln* (*z* / *of_nat k* + *1*))

**theorem** *ln_Gamma_complex_LIMSEQ*: (*z* :: *complex*) ∉ $\mathbb{Z}_{\le 0}$ ⟹ *ln_Gamma_series z* ⟶ *ln_Gamma z*

### 6.20.2 The Polygamma functions

**definition** *Polygamma* :: *nat* ⇒ (*'a* :: {*real_normed_field*,*banach*}) ⇒ *'a* **where**
*Polygamma n z* = (*if n* = *0 then*
($\sum$ *k*. *inverse* (*of_nat* (*Suc k*)) − *inverse* (*z* + *of_nat k*)) − *euler_mascheroni* *else*
(−*1*)^*Suc n* ∗ *fact n* ∗ ($\sum$ *k*. *inverse* ((*z* + *of_nat k*)^*Suc n*)))

**abbreviation** *Digamma* :: (*'a* :: {*real_normed_field*,*banach*}) ⇒ *'a* **where**
*Digamma* ≡ *Polygamma 0*

**theorem** *Digamma_LIMSEQ*:
**fixes** *z* :: *'a* :: {*banach*,*real_normed_field*}
**assumes** *z*: *z* ≠ *0*
**shows** (λ*m*. *of_real* (*ln* (*real m*)) − ($\sum$ *n*<*m*. *inverse* (*z* + *of_nat n*))) ⟶ *Digamma z*

**theorem** *Polygamma_LIMSEQ*:
**fixes** *z* :: *'a* :: {*banach*,*real_normed_field*}
**assumes** *z* ≠ *0* **and** *n* > *0*
**shows** (λ*k*. *inverse* ((*z* + *of_nat k*)^*Suc n*)) *sums* ((−*1*) ^ *Suc n* ∗ *Polygamma n z* / *fact n*)

**theorem** *has_field_derivative_ln_Gamma_complex* [*derivative_intros*]:
**fixes** *z* :: *complex*
**assumes** *z*: *z* ∉ $\mathbb{R}_{\leq 0}$
**shows** (*ln_Gamma has_field_derivative Digamma z*) (*at z*)

**theorem** *Polygamma_plus1*:
**assumes** *z* ≠ *0*
**shows** *Polygamma n* (*z* + *1*) = *Polygamma n z* + (−*1*)^*n* ∗ *fact n* / (*z* ^ *Suc n*)

**theorem** *Digamma_of_nat*:
*Digamma* (*of_nat* (*Suc n*) :: *'a* :: {*real_normed_field*,*banach*}) = *harm n* − *euler_mascheroni*

**theorem** *has_field_derivative_Polygamma* [*derivative_intros*]:
**fixes** *z* :: *'a* :: {*real_normed_field*,*euclidean_space*}
**assumes** *z*: *z* ∉ $\mathbb{Z}_{\leq 0}$
**shows** (*Polygamma n has_field_derivative Polygamma* (*Suc n*) *z*) (*at z within A*)

### 6.20.3 Basic properties

**theorem** *Gamma_series_LIMSEQ* [*tendsto_intros*]:

*Gamma_series z* $\longrightarrow$ *Gamma z*

**theorem** *Gamma_plus1*: $z \notin \mathbb{Z}_{\leq 0} \Longrightarrow$ *Gamma* $(z + 1) = z *$ *Gamma z*

**theorem** *pochhammer_Gamma*: $z \notin \mathbb{Z}_{\leq 0} \Longrightarrow$ *pochhammer z n* = *Gamma* (*z* + *of_nat n*) / *Gamma z*

**theorem** *Gamma_fact*: *Gamma* (*1* + *of_nat n*) = *fact n*

### 6.20.4 Differentiability

**theorem** *has_field_derivative_Gamma* [*derivative_intros*]:
$z \notin \mathbb{Z}_{\leq 0} \Longrightarrow$ (*Gamma has_field_derivative Gamma z* ∗ *Digamma z*) (*at z within A*)

**theorem** *log_convex_Gamma_real*: *convex_on* {*0*<..} (*ln* ∘ *Gamma* :: *real* ⇒ *real*)

### 6.20.5 The uniqueness of the real Gamma function

**theorem** *Gamma_pos_real_unique*:
**assumes** *x*: $x > 0$
**shows** *G x* = *Gamma x*

### 6.20.6 The Beta function

**theorem** *Beta_plus1_plus1*:
**assumes** $x \notin \mathbb{Z}_{\leq 0}$ $y \notin \mathbb{Z}_{\leq 0}$
**shows** *Beta* (*x* + *1*) *y* + *Beta x* (*y* + *1*) = *Beta x y*

**theorem** *Beta_plus1_left*:
**assumes** $x \notin \mathbb{Z}_{\leq 0}$
**shows** (*x* + *y*) ∗ *Beta* (*x* + *1*) *y* = *x* ∗ *Beta x y*

**theorem** *Beta_plus1_right*:
**assumes** $y \notin \mathbb{Z}_{\leq 0}$
**shows** (*x* + *y*) ∗ *Beta x* (*y* + *1*) = *y* ∗ *Beta x y*

### 6.20.7 Legendre duplication theorem

**theorem** *Gamma_legendre_duplication*:
**fixes** *z* :: *complex*
**assumes** $z \notin \mathbb{Z}_{\leq 0}$ $z + 1/2 \notin \mathbb{Z}_{\leq 0}$
**shows** *Gamma z* ∗ *Gamma* (*z* + *1/2*) =
*exp* ((*1* − *2*∗*z*) ∗ *of_real* (*ln 2*)) ∗ *of_real* (*sqrt pi*) ∗ *Gamma* (*2*∗*z*)

### 6.20.8 Alternative definitions

**theorem** *Gamma_series_euler'*:
**assumes** *z*: (*z* :: *'a* :: *Gamma*) $\notin \mathbb{Z}_{\leq 0}$
**shows** (λ*n*. *Gamma_series_euler' z n*) ⟶ *Gamma z*

**theorem** *Gamma_Weierstrass_complex*: *Gamma_series_Weierstrass z* ⟶ *Gamma* (*z* :: *complex*)

**theorem** *gbinomial_Gamma*:
**assumes** $z + 1 \notin \mathbb{Z}_{\leq 0}$
**shows** (*z gchoose n*) = *Gamma* (*z* + *1*) / (*fact n* ∗ *Gamma* (*z* − *of_nat n* + *1*))

**theorem** *Gamma_integral_complex*:
**assumes** *z*: *Re z* > *0*
**shows** ((λ*t*. *of_real t powr* (*z* − *1*) / *of_real* (*exp t*)) *has_integral Gamma z*) {*0*..}

**theorem** *has_integral_Beta_real*:
**assumes** *a*: *a* > *0* **and** *b*: *b* > (*0* :: *real*)
**shows** ((λ*t*. *t powr* (*a* − *1*) ∗ (*1* − *t*) *powr* (*b* − *1*)) *has_integral Beta a b*) {*0*..*1*}

### 6.20.9 The Weierstraß product formula for the sine

**theorem** *sin_product_formula_complex*:
**fixes** *z* :: *complex*
**shows** (λ*n*. *of_real pi* ∗ *z* ∗ (∏*k*=*1*..*n*. *1* − *z*^*2* / *of_nat k*^*2*)) ⟶ *sin* (*of_real pi* ∗ *z*)

**theorem** *wallis*: (λ*n*. ∏*k*=*1*..*n*. (*4*∗*real k*^*2*) / (*4*∗*real k*^*2* − *1*)) ⟶ *pi* / *2*

### 6.20.10 The Solution to the Basel problem

**theorem** *inverse_squares_sums*: $(\lambda n.\ 1\ /\ (n + 1)^2)$ *sums* $(pi^2\ /\ 6)$

**end**

**theory** *Interval_Integral*
  **imports** *Equivalence_Lebesgue_Henstock_Integration*
**begin**

### 6.20.11 Approximating a (possibly infinite) interval

**proposition** *einterval_Icc_approximation*:
  **fixes** *a b* :: *ereal*
  **assumes** $a < b$
  **obtains** *u l* :: *nat* ⇒ *real* **where**
    *einterval a b* = $(\bigcup i.\ \{l\ i\ ..\ u\ i\})$
    *incseq u decseq l* $\bigwedge i.\ l\ i < u\ i$ $\bigwedge i.\ a < l\ i$ $\bigwedge i.\ u\ i < b$
    $l \longrightarrow a$ $u \longrightarrow b$

**definition** *interval_lebesgue_integral* :: *real measure* ⇒ *ereal* ⇒ *ereal* ⇒ (*real* ⇒ *'a*) ⇒ *'a*::{*banach*, *second_countable_topology*} **where**
  *interval_lebesgue_integral M a b f* =
    (*if* $a \le b$ *then* (*LINT x*:*einterval a b*|*M*. *f x*) *else* − (*LINT x*:*einterval b a*|*M*. *f x*))

**definition** *interval_lebesgue_integrable* :: *real measure* ⇒ *ereal* ⇒ *ereal* ⇒ (*real* ⇒ *'a*::{*banach*, *second_countable_topology*}) ⇒ *bool* **where**
  *interval_lebesgue_integrable M a b f* =
    (*if* $a \le b$ *then set_integrable M* (*einterval a b*) *f else set_integrable M* (*einterval b a*) *f*)

### 6.20.12 Basic properties of integration over an interval

**proposition** *interval_integrable_to_infinity_eq*: (*interval_lebesgue_integrable M a* ∞ *f*) =
  (*set_integrable M* {*a*<..} *f*)

### 6.20.13 Basic properties of integration over an interval wrt lebesgue measure

### 6.20.14 General limit approximation arguments

**proposition** *interval_integral_Icc_approx_nonneg*:
**fixes** $a\ b$ :: *ereal*
**assumes** $a < b$
**fixes** $u\ l$ :: *nat* ⇒ *real*
**assumes** *approx*: *einterval a b* = $(\bigcup i.\ \{l\ i\ ..\ u\ i\})$
*incseq u decseq l* $\bigwedge i.\ l\ i < u\ i\ \bigwedge i.\ a < l\ i\ \bigwedge i.\ u\ i < b$
$l \longrightarrow a\ u \longrightarrow b$
**fixes** $f$ :: *real* ⇒ *real*
**assumes** *f_integrable*: $\bigwedge i.$ *set_integrable lborel* $\{l\ i..u\ i\}\ f$
**assumes** *f_nonneg*: *AE x in lborel. a < ereal x* ⟶ *ereal x < b* ⟶ $0 \le f\ x$
**assumes** *f_measurable*: *set_borel_measurable lborel* (*einterval a b*) *f*
**assumes** *lbint_lim*: (λ*i*. *LBINT x=l i.. u i. f x*) ⟶ *C*
**shows**
*set_integrable lborel* (*einterval a b*) *f*
(*LBINT x=a..b. f x*) = *C*

**proposition** *interval_integral_Icc_approx_integrable*:
**fixes** $u\ l$ :: *nat* ⇒ *real* **and** $a\ b$ :: *ereal*
**fixes** $f$ :: *real* ⇒ *'a*::{*banach*, *second_countable_topology*}
**assumes** $a < b$
**assumes** *approx*: *einterval a b* = $(\bigcup i.\ \{l\ i\ ..\ u\ i\})$
*incseq u decseq l* $\bigwedge i.\ l\ i < u\ i\ \bigwedge i.\ a < l\ i\ \bigwedge i.\ u\ i < b$
$l \longrightarrow a\ u \longrightarrow b$
**assumes** *f_integrable*: *set_integrable lborel* (*einterval a b*) *f*
**shows** (λ*i*. *LBINT x=l i.. u i. f x*) ⟶ (*LBINT x=a..b. f x*)

### 6.20.15 A slightly stronger Fundamental Theorem of Calculus

**theorem** *interval_integral_FTC_integrable*:
**fixes** $f\ F$ :: *real* ⇒ *'a*::*euclidean_space* **and** $a\ b$ :: *ereal*
**assumes** $a < b$
**assumes** *F*: $\bigwedge x.$ *a < ereal x* ⟹ *ereal x < b* ⟹ (*F has_vector_derivative f x*) (*at x*)
**assumes** *f*: $\bigwedge x.$ *a < ereal x* ⟹ *ereal x < b* ⟹ *isCont f x*
**assumes** *f_integrable*: *set_integrable lborel* (*einterval a b*) *f*
**assumes** *A*: ((*F* ∘ *real_of_ereal*) ⟶ *A*) (*at_right a*)
**assumes** *B*: ((*F* ∘ *real_of_ereal*) ⟶ *B*) (*at_left b*)
**shows** (*LBINT x=a..b. f x*) = *B* − *A*

**theorem** *interval_integral_FTC2*:
**fixes** $a\ b\ c$ :: *real* **and** $f$ :: *real* ⇒ *'a::euclidean_space*
**assumes** $a \le c$ $c \le b$
**and** *contf*: *continuous_on* $\{a..b\}$ $f$
**fixes** $x$ :: *real*
**assumes** $a \le x$ **and** $x \le b$
**shows** $((\lambda u.$ *LBINT* $y{=}c..u.$ $f\ y)$ *has_vector_derivative* $(f\ x))$ (*at* $x$ *within* $\{a..b\}$)

**proposition** *einterval_antiderivative*:
**fixes** $a\ b$ :: *ereal* **and** $f$ :: *real* ⇒ *'a::euclidean_space*
**assumes** $a < b$ **and** *contf*: $\bigwedge x$ :: *real*. $a < x \Longrightarrow x < b \Longrightarrow$ *isCont* $f\ x$
**shows** $\exists F.\ \forall x$ :: *real*. $a < x \longrightarrow x < b \longrightarrow$ ($F$ *has_vector_derivative* $f\ x$) (*at* $x$)

### 6.20.16 The substitution theorem

**theorem** *interval_integral_substitution_finite*:
**fixes** $a\ b$ :: *real* **and** $f$ :: *real* ⇒ *'a::euclidean_space*
**assumes** $a \le b$
**and** *derivg*: $\bigwedge x.\ a \le x \Longrightarrow x \le b \Longrightarrow$ ($g$ *has_real_derivative* $(g'\ x)$) (*at* $x$ *within* $\{a..b\}$)
**and** *contf* : *continuous_on* ($g$ ` $\{a..b\}$) $f$
**and** *contg'*: *continuous_on* $\{a..b\}$ $g'$
**shows** *LBINT* $x{=}a..b.$ $g'\ x *_R f\ (g\ x)$ = *LBINT* $y{=}g\ a..g\ b.$ $f\ y$

**theorem** *interval_integral_substitution_integrable*:
**fixes** $f$ :: *real* ⇒ *'a::euclidean_space* **and** $a\ b\ u\ v$ :: *ereal*
**assumes** $a < b$
**and** *deriv_g*: $\bigwedge x.\ a <$ *ereal* $x \Longrightarrow$ *ereal* $x < b \Longrightarrow$ *DERIV* $g\ x$ :> $g'\ x$
**and** *contf*: $\bigwedge x.\ a <$ *ereal* $x \Longrightarrow$ *ereal* $x < b \Longrightarrow$ *isCont* $f\ (g\ x)$
**and** *contg'*: $\bigwedge x.\ a <$ *ereal* $x \Longrightarrow$ *ereal* $x < b \Longrightarrow$ *isCont* $g'\ x$
**and** *g'_nonneg*: $\bigwedge x.\ a \le$ *ereal* $x \Longrightarrow$ *ereal* $x \le b \Longrightarrow 0 \le g'\ x$
**and** $A$: ((*ereal* ∘ $g$ ∘ *real_of_ereal*) ⟶ $A$) (*at_right* $a$)
**and** $B$: ((*ereal* ∘ $g$ ∘ *real_of_ereal*) ⟶ $B$) (*at_left* $b$)
**and** *integrable*: *set_integrable lborel* (*einterval* $a\ b$) $(\lambda x.\ g'\ x *_R f\ (g\ x))$
**and** *integrable2*: *set_integrable lborel* (*einterval* $A\ B$) $(\lambda x.\ f\ x)$
**shows** (*LBINT* $x{=}A..B.$ $f\ x$) = (*LBINT* $x{=}a..b.$ $g'\ x *_R f\ (g\ x)$)

**theorem** *interval_integral_substitution_nonneg*:
**fixes** $f\ g\ g'$:: *real* ⇒ *real* **and** $a\ b\ u\ v$ :: *ereal*
**assumes** $a < b$
**and** *deriv_g*: $\bigwedge x.\ a <$ *ereal* $x \Longrightarrow$ *ereal* $x < b \Longrightarrow$ *DERIV* $g\ x$ :> $g'\ x$
**and** *contf*: $\bigwedge x.\ a <$ *ereal* $x \Longrightarrow$ *ereal* $x < b \Longrightarrow$ *isCont* $f\ (g\ x)$
**and** *contg'*: $\bigwedge x.\ a <$ *ereal* $x \Longrightarrow$ *ereal* $x < b \Longrightarrow$ *isCont* $g'\ x$
**and** *f_nonneg*: $\bigwedge x.\ a <$ *ereal* $x \Longrightarrow$ *ereal* $x < b \Longrightarrow 0 \le f\ (g\ x)$

**and** *g′_nonneg*: $\bigwedge x.\ a \leq ereal\ x \Longrightarrow ereal\ x \leq b \Longrightarrow 0 \leq g'\ x$
**and** *A*: $((ereal \circ g \circ real\_of\_ereal) \longrightarrow A)\ (at\_right\ a)$
**and** *B*: $((ereal \circ g \circ real\_of\_ereal) \longrightarrow B)\ (at\_left\ b)$
**and** *integrable_fg*: $set\_integrable\ lborel\ (einterval\ a\ b)\ (\lambda x.\ f\ (g\ x) * g'\ x)$
**shows**
*set_integrable lborel* (*einterval A B*) *f*
$(LBINT\ x{=}A..B.\ f\ x) = (LBINT\ x{=}a..b.\ (f\ (g\ x) * g'\ x))$

**proposition** *interval_integral_norm*:
**fixes** $f :: real \Rightarrow 'a :: \{banach,\ second\_countable\_topology\}$
**shows** $interval\_lebesgue\_integrable\ lborel\ a\ b\ f \Longrightarrow a \leq b \Longrightarrow$
$norm\ (LBINT\ t{=}a..b.\ f\ t) \leq LBINT\ t{=}a..b.\ norm\ (f\ t)$

**proposition** *interval_integral_norm2*:
$interval\_lebesgue\_integrable\ lborel\ a\ b\ f \Longrightarrow$
$norm\ (LBINT\ t{=}a..b.\ f\ t) \leq |LBINT\ t{=}a..b.\ norm\ (f\ t)|$

**end**

## 6.21 Integration by Substition for the Lebesgue Integral

**theory** *Lebesgue_Integral_Substitution*
**imports** *Interval_Integral*
**begin**

**theorem** *nn_integral_substitution*:
**fixes** $f :: real \Rightarrow real$
**assumes** *Mf*[*measurable*]: $set\_borel\_measurable\ borel\ \{g\ a..g\ b\}\ f$
**assumes** *derivg*: $\bigwedge x.\ x \in \{a..b\} \Longrightarrow (g\ has\_real\_derivative\ g'\ x)\ (at\ x)$
**assumes** *contg′*: $continuous\_on\ \{a..b\}\ g'$
**assumes** *derivg_nonneg*: $\bigwedge x.\ x \in \{a..b\} \Longrightarrow g'\ x \geq 0$
**assumes** $a \leq b$
**shows** $(\int^{+} x.\ f\ x * indicator\ \{g\ a..g\ b\}\ x\ \partial lborel) =$
$(\int^{+} x.\ f\ (g\ x) * g'\ x * indicator\ \{a..b\}\ x\ \partial lborel)$

**theorem** *integral_substitution*:
**assumes** *integrable*: $set\_integrable\ lborel\ \{g\ a..g\ b\}\ f$
**assumes** *derivg*: $\bigwedge x.\ x \in \{a..b\} \Longrightarrow (g\ has\_real\_derivative\ g'\ x)\ (at\ x)$
**assumes** *contg′*: $continuous\_on\ \{a..b\}\ g'$
**assumes** *derivg_nonneg*: $\bigwedge x.\ x \in \{a..b\} \Longrightarrow g'\ x \geq 0$
**assumes** $a \leq b$
**shows** $set\_integrable\ lborel\ \{a..b\}\ (\lambda x.\ f\ (g\ x) * g'\ x)$
**and** $(LBINT\ x.\ f\ x * indicator\ \{g\ a..g\ b\}\ x) = (LBINT\ x.\ f\ (g\ x) * g'\ x * indicator\ \{a..b\}\ x)$

**theorem** *interval_integral_substitution*:
**assumes** *integrable*: *set_integrable lborel* {*g a*..*g b*} *f*
**assumes** *derivg*: $\bigwedge x.\ x \in \{a..b\} \Longrightarrow (g\ has\_real\_derivative\ g'\ x)\ (at\ x)$
**assumes** *contg'*: *continuous_on* {*a*..*b*} *g'*
**assumes** *derivg_nonneg*: $\bigwedge x.\ x \in \{a..b\} \Longrightarrow g'\ x \geq 0$
**assumes** $a \leq b$
**shows** *set_integrable lborel* {*a*..*b*} ($\lambda x.\ f\ (g\ x) * g'\ x$)
**and** (*LBINT x*=*g a*..*g b*. *f x*) = (*LBINT x*=*a*..*b*. *f* (*g x*) ∗ *g' x*)

**end**

## 6.22 The Volume of an $n$-Dimensional Ball

**theory** *Ball_Volume*
**imports** *Gamma_Function Lebesgue_Integral_Substitution*
**begindefinition** *unit_ball_vol* :: *real* ⇒ *real* **where**
*unit_ball_vol n* = *pi powr* (*n* / *2*) / *Gamma* (*n* / *2* + *1*)

**corollary** *content_ball*:
*content* (*ball c r*) = *unit_ball_vol* (*DIM*('*a*)) ∗ *r* ^ *DIM*('*a*)

**end**

## 6.23 Integral Test for Summability

**theory** *Integral_Test*
**imports** *Henstock_Kurzweil_Integration*
**beginlocale** *antimono_fun_sum_integral_diff* =
**fixes** *f* :: *real* ⇒ *real*
**assumes** *dec*: $\bigwedge x\ y.\ x \geq 0 \Longrightarrow x \leq y \Longrightarrow f\ x \geq f\ y$
**assumes** *nonneg*: $\bigwedge x.\ x \geq 0 \Longrightarrow f\ x \geq 0$
**assumes** *cont*: *continuous_on* {*0*..} *f*
**begin**

**theorem** *integral_test*:
*summable* ($\lambda n$. *f* (*of_nat n*)) ⟷ *convergent* ($\lambda n$. *integral* {*0*..*of_nat n*} *f*)

**end**

## 6.24 Continuity of the indefinite integral; improper integral theorem

**theory** *Improper_Integral*
**imports** *Equivalence_Lebesgue_Henstock_Integration*
**begin**

### 6.24.1 Equiintegrability

**definition** *equiintegrable_on* (**infixr** *equiintegrable′_on 46*)
**where** $F$ *equiintegrable_on* $I \equiv$
$(\forall f \in F.\ f\ \mathit{integrable\_on}\ I) \wedge$
$(\forall e > 0.\ \exists \gamma.\ \mathit{gauge}\ \gamma \wedge$
$(\forall f\ \mathcal{D}.\ f \in F \wedge \mathcal{D}\ \mathit{tagged\_division\_of}\ I \wedge \gamma\ \mathit{fine}\ \mathcal{D}$
$\longrightarrow \mathit{norm}\ ((\sum (x,K) \in \mathcal{D}.\ \mathit{content}\ K *_R f\ x) - \mathit{integral}\ I\ f)$
$< e))$

**corollary** *equiintegrable_sum_real*:
**fixes** $F :: (\mathit{real} \Rightarrow {'b}::\mathit{euclidean\_space})\ \mathit{set}$
**assumes** $F$ *equiintegrable_on* $\{a..b\}$
**shows** $(\bigcup I \in \mathit{Collect\ finite}.\ \bigcup c \in \{c.\ (\forall i \in I.\ c\ i \geq 0) \wedge \mathit{sum}\ c\ I = 1\}.$
$\bigcup f \in I \rightarrow F.\ \{(\lambda x.\ \mathit{sum}\ (\lambda i.\ c\ i *_R f\ i\ x)\ I)\})$
*equiintegrable_on* $\{a..b\}$

**theorem** *equiintegrable_limit*:
**fixes** $g :: {'a} :: \mathit{euclidean\_space} \Rightarrow {'b} :: \mathit{banach}$
**assumes** *feq*: *range f equiintegrable_on cbox a b*
**and** *to_g*: $\bigwedge x.\ x \in \mathit{cbox}\ a\ b \Longrightarrow (\lambda n.\ f\ n\ x) \longrightarrow g\ x$
**shows** $g$ *integrable_on cbox a b* $\wedge\ (\lambda n.\ \mathit{integral}\ (\mathit{cbox}\ a\ b)\ (f\ n)) \longrightarrow \mathit{integral}$
$(\mathit{cbox}\ a\ b)\ g$

### 6.24.2 Subinterval restrictions for equiintegrable families

**proposition** *sum_content_area_over_thin_division*:
**assumes** *div*: $\mathcal{D}$ *division_of* $S$ **and** $S$: $S \subseteq \mathit{cbox}\ a\ b$ **and** $i$: $i \in \mathit{Basis}$
**and** $a \cdot i \leq c\ c \leq b \cdot i$
**and** *nonmt*: $\bigwedge K.\ K \in \mathcal{D} \Longrightarrow K \cap \{x.\ x \cdot i = c\} \neq \{\}$
**shows** $(b \cdot i - a \cdot i) * (\sum K{\in}\mathcal{D}.\ \mathit{content}\ K\ /\ (\mathit{interval\_upperbound}\ K \cdot i -$
$\mathit{interval\_lowerbound}\ K \cdot i))$
$\leq 2 * \mathit{content}(\mathit{cbox}\ a\ b)$

**proposition** *bounded_equiintegral_over_thin_tagged_partial_division*:
**fixes** $f :: {'a}::\mathit{euclidean\_space} \Rightarrow {'b}::\mathit{euclidean\_space}$
**assumes** $F$: $F$ *equiintegrable_on cbox a b* **and** $f$: $f \in F$ **and** $0 < \varepsilon$
**and** *norm_f*: $\bigwedge h\ x.\ [\![h \in F;\ x \in \mathit{cbox}\ a\ b]\!] \Longrightarrow \mathit{norm}(h\ x) \leq \mathit{norm}(f\ x)$
**obtains** $\gamma$ **where** *gauge* $\gamma$
$\bigwedge c\ i\ S\ h.\ [\![c \in \mathit{cbox}\ a\ b;\ i \in \mathit{Basis};\ S\ \mathit{tagged\_partial\_division\_of}\ \mathit{cbox}\ a$
$b;$

$\gamma$ *fine* $S$; $h \in F$; $\bigwedge x\ K.\ (x,K) \in S \Longrightarrow (K \cap \{x.\ x \cdot i = c \cdot i\}$
$\neq \{\})]\!]$
$\Longrightarrow (\sum (x,K) \in S.\ norm\ (integral\ K\ h)) < \varepsilon$

**proposition** *equiintegrable_halfspace_restrictions_le*:
**fixes** $f$ :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
**assumes** $F$: $F$ *equiintegrable_on cbox a b* **and** $f$: $f \in F$
**and** *norm_f*: $\bigwedge h\ x.\ [\![h \in F; x \in cbox\ a\ b]\!] \Longrightarrow norm(h\ x) \leq norm(f\ x)$
**shows** $(\bigcup i \in Basis.\ \bigcup c.\ \bigcup h \in F.\ \{(\lambda x.\ if\ x \cdot i \leq c\ then\ h\ x\ else\ 0)\})$
*equiintegrable_on cbox a b*

**corollary** *equiintegrable_halfspace_restrictions_ge*:
**fixes** $f$ :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
**assumes** $F$: $F$ *equiintegrable_on cbox a b* **and** $f$: $f \in F$
**and** *norm_f*: $\bigwedge h\ x.\ [\![h \in F; x \in cbox\ a\ b]\!] \Longrightarrow norm(h\ x) \leq norm(f\ x)$
**shows** $(\bigcup i \in Basis.\ \bigcup c.\ \bigcup h \in F.\ \{(\lambda x.\ if\ x \cdot i \geq c\ then\ h\ x\ else\ 0)\})$
*equiintegrable_on cbox a b*

**corollary** *equiintegrable_halfspace_restrictions_lt*:
**fixes** $f$ :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
**assumes** $F$: $F$ *equiintegrable_on cbox a b* **and** $f$: $f \in F$
**and** *norm_f*: $\bigwedge h\ x.\ [\![h \in F; x \in cbox\ a\ b]\!] \Longrightarrow norm(h\ x) \leq norm(f\ x)$
**shows** $(\bigcup i \in Basis.\ \bigcup c.\ \bigcup h \in F.\ \{(\lambda x.\ if\ x \cdot i < c\ then\ h\ x\ else\ 0)\})$ *equiintegrable_on cbox a b*
(**is** *?G equiintegrable_on cbox a b*)

**corollary** *equiintegrable_halfspace_restrictions_gt*:
**fixes** $f$ :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
**assumes** $F$: $F$ *equiintegrable_on cbox a b* **and** $f$: $f \in F$
**and** *norm_f*: $\bigwedge h\ x.\ [\![h \in F; x \in cbox\ a\ b]\!] \Longrightarrow norm(h\ x) \leq norm(f\ x)$
**shows** $(\bigcup i \in Basis.\ \bigcup c.\ \bigcup h \in F.\ \{(\lambda x.\ if\ x \cdot i > c\ then\ h\ x\ else\ 0)\})$ *equiintegrable_on cbox a b*
(**is** *?G equiintegrable_on cbox a b*)

**proposition** *equiintegrable_closed_interval_restrictions*:
**fixes** $f$ :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
**assumes** $f$: $f$ *integrable_on cbox a b*
**shows** $(\bigcup c\ d.\ \{(\lambda x.\ if\ x \in cbox\ c\ d\ then\ f\ x\ else\ 0)\})$ *equiintegrable_on cbox a b*

### 6.24.3 Continuity of the indefinite integral

**proposition** *indefinite_integral_continuous*:
**fixes** $f$ :: *'a :: euclidean_space* ⇒ *'b :: euclidean_space*
**assumes** *int_f*: $f$ *integrable_on cbox a b*
**and** $c$: $c \in cbox\ a\ b$ **and** $d$: $d \in cbox\ a\ b$ $0 < \varepsilon$

**obtains** $\delta$ **where** $0 < \delta$
$\bigwedge c'\ d'.\ [\![c' \in cbox\ a\ b;\ d' \in cbox\ a\ b;\ norm(c' - c) \le \delta;\ norm(d' - d) \le \delta]\!]$
$\Longrightarrow norm(integral(cbox\ c'\ d')\ f - integral(cbox\ c\ d)\ f) < \varepsilon$

**corollary** *indefinite_integral_uniformly_continuous*:
**fixes** $f :: 'a :: euclidean\_space \Rightarrow 'b :: euclidean\_space$
**assumes** $f$ *integrable_on cbox a b*
**shows** *uniformly_continuous_on* $(cbox\ (Pair\ a\ a)\ (Pair\ b\ b))\ (\lambda y.\ integral\ (cbox\ (fst\ y)\ (snd\ y))\ f)$

**corollary** *bounded_integrals_over_subintervals*:
**fixes** $f :: 'a :: euclidean\_space \Rightarrow 'b :: euclidean\_space$
**assumes** $f$ *integrable_on cbox a b*
**shows** *bounded* $\{integral\ (cbox\ c\ d)\ f\ |c\ d.\ cbox\ c\ d \subseteq cbox\ a\ b\}$
**theorem** *absolutely_integrable_improper*:
**fixes** $f :: 'M::euclidean\_space \Rightarrow 'N::euclidean\_space$
**assumes** *int_f*: $\bigwedge c\ d.\ cbox\ c\ d \subseteq box\ a\ b \Longrightarrow f\ integrable\_on\ cbox\ c\ d$
**and** *bo*: $bounded\ \{integral\ (cbox\ c\ d)\ f\ |c\ d.\ cbox\ c\ d \subseteq box\ a\ b\}$
**and** *absi*: $\bigwedge i.\ i \in Basis$
$\Longrightarrow \exists g.\ g\ absolutely\_integrable\_on\ cbox\ a\ b\ \wedge$
$((\forall x \in cbox\ a\ b.\ f\ x \cdot i \le g\ x) \vee (\forall x \in cbox\ a\ b.\ f\ x \cdot i \ge g\ x))$
**shows** $f$ *absolutely_integrable_on cbox a b*

### 6.24.4 Second mean value theorem and corollaries

**theorem** *second_mean_value_theorem_full*:
**fixes** $f :: real \Rightarrow real$
**assumes** $f$: $f\ integrable\_on\ \{a..b\}$ **and** $a \le b$
**and** $g$: $\bigwedge x\ y.\ [\![a \le x;\ x \le y;\ y \le b]\!] \Longrightarrow g\ x \le g\ y$
**obtains** $c$ **where** $c \in \{a..b\}$
**and** $((\lambda x.\ g\ x * f\ x)\ has\_integral\ (g\ a * integral\ \{a..c\}\ f + g\ b * integral\ \{c..b\}\ f))\ \{a..b\}$

**corollary** *second_mean_value_theorem*:
**fixes** $f :: real \Rightarrow real$
**assumes** $f$: $f\ integrable\_on\ \{a..b\}$ **and** $a \le b$
**and** $g$: $\bigwedge x\ y.\ [\![a \le x;\ x \le y;\ y \le b]\!] \Longrightarrow g\ x \le g\ y$
**obtains** $c$ **where** $c \in \{a..b\}$
$integral\ \{a..b\}\ (\lambda x.\ g\ x * f\ x) = g\ a * integral\ \{a..c\}\ f + g\ b * integral\ \{c..b\}\ f$

**end**

## 6.25 Continuous Extensions of Functions

**theory** *Continuous_Extension*
**imports** *Starlike*
**begin**

### 6.25.1 Partitions of unity subordinate to locally finite open coverings

**proposition** *subordinate_partition_of_unity*:
  **fixes** $S$ :: *'a::metric_space set*
  **assumes** $S \subseteq \bigcup\mathcal{C}$ **and** *opC*: $\bigwedge T.\ T \in \mathcal{C} \Longrightarrow$ *open* $T$
    **and** *fin*: $\bigwedge x.\ x \in S \Longrightarrow \exists V.$ *open* $V \wedge x \in V \wedge$ *finite* $\{U \in \mathcal{C}.\ U \cap V \neq \{\}\}$
  **obtains** $F$ :: $['a\ set,\ 'a] \Rightarrow real$
    **where** $\bigwedge U.\ U \in \mathcal{C} \Longrightarrow$ *continuous_on* $S$ $(F\ U) \wedge (\forall x \in S.\ 0 \leq F\ U\ x)$
      **and** $\bigwedge x\ U.\ [\![U \in \mathcal{C};\ x \in S;\ x \notin U]\!] \Longrightarrow F\ U\ x = 0$
      **and** $\bigwedge x.\ x \in S \Longrightarrow$ *supp_sum* $(\lambda W.\ F\ W\ x)\ \mathcal{C} = 1$
      **and** $\bigwedge x.\ x \in S \Longrightarrow \exists V.$ *open* $V \wedge x \in V \wedge$ *finite* $\{U \in \mathcal{C}.\ \exists x{\in}V.\ F\ U\ x \neq 0\}$

### 6.25.2 Urysohn's Lemma for Euclidean Spaces

**proposition** *Urysohn_local_strong*:
  **assumes** *US*: *closedin* (*top_of_set* $U$) $S$
    **and** *UT*: *closedin* (*top_of_set* $U$) $T$
    **and** $S \cap T = \{\}$ $a \neq b$
  **obtains** $f$ :: *'a::euclidean_space* $\Rightarrow$ *'b::euclidean_space*
    **where** *continuous_on* $U$ $f$
      $\bigwedge x.\ x \in U \Longrightarrow f\ x \in$ *closed_segment* $a$ $b$
      $\bigwedge x.\ x \in U \Longrightarrow (f\ x = a \longleftrightarrow x \in S)$
      $\bigwedge x.\ x \in U \Longrightarrow (f\ x = b \longleftrightarrow x \in T)$

**proposition** *Urysohn*:
  **assumes** *US*: *closed* $S$
    **and** *UT*: *closed* $T$
    **and** $S \cap T = \{\}$
  **obtains** $f$ :: *'a::euclidean_space* $\Rightarrow$ *'b::euclidean_space*
    **where** *continuous_on UNIV* $f$
      $\bigwedge x.\ f\ x \in$ *closed_segment* $a$ $b$
      $\bigwedge x.\ x \in S \Longrightarrow f\ x = a$
      $\bigwedge x.\ x \in T \Longrightarrow f\ x = b$

### 6.25.3 Dugundji's Extension Theorem and Tietze Variants

**theorem** *Dugundji*:
  **fixes** $f$ :: $'a$::{*metric_space*,*second_countable_topology*} ⇒ $'b$::*real_inner*
  **assumes** *convex* $C$ $C \neq \{\}$
    **and** *cloin*: *closedin* (*top_of_set* $U$) $S$
    **and** *contf*: *continuous_on* $S$ $f$ **and** $f$ ‘ $S \subseteq C$
  **obtains** $g$ **where** *continuous_on* $U$ $g$ $g$ ‘ $U \subseteq C$
    $\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$

**corollary** *Tietze*:
  **fixes** $f$ :: $'a$::{*metric_space*,*second_countable_topology*} ⇒ $'b$::*real_inner*
  **assumes** *continuous_on* $S$ $f$
   **and** *closedin* (*top_of_set* $U$) $S$
   **and** $0 \leq B$
   **and** $\bigwedge x.\ x \in S \Longrightarrow norm(f\ x) \leq B$
  **obtains** $g$ **where** *continuous_on* $U$ $g$ $\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$
   $\bigwedge x.\ x \in U \Longrightarrow norm(g\ x) \leq B$

**end**

## 6.26 Equivalence Between Classical Borel Measurability and HOL Light's

**theory** *Equivalence_Measurable_On_Borel*
  **imports** *Equivalence_Lebesgue_Henstock_Integration Improper_Integral Continuous_Extension*
**begin**

### 6.26.1 Austin's Lemma

### 6.26.2 A differentiability-like property of the indefinite integral.

**proposition** *integrable_ccontinuous_explicit*:
  **fixes** $f$ :: $'a$::*euclidean_space* ⇒ $'b$::*euclidean_space*
  **assumes** $\bigwedge a\ b$::$'a$. $f$ *integrable_on cbox a b*
  **obtains** $N$ **where**
    *negligible* $N$
    $\bigwedge x\ e.\ [\![x \notin N;\ 0 < e]\!] \Longrightarrow$
      $\exists d>0.\ \forall h.\ 0 < h \wedge h < d \longrightarrow$
        $norm(integral\ (cbox\ x\ (x + h *_R One))\ f\ /_R\ h\ \hat{}\ DIM('a) - f\ x) < e$

### 6.26.3 HOL Light measurability

**proposition** *integrable_subintervals_imp_measurable*:
**fixes** $f$ :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
**assumes** $\bigwedge a\ b.$ *f integrable_on cbox a b*
**shows** *f measurable_on UNIV*

### 6.26.4 Composing continuous and measurable functions; a few variants

**proposition** *indicator_measurable_on*:
**assumes** $S \in$ *sets lebesgue*
**shows** *indicat_real S measurable_on UNIV*

**lemma** *simple_function_induct_real*
[*consumes 1*, *case_names cong set mult add*, *induct set*: *simple_function*]:
**fixes** $u$ :: *'a* ⇒ *real*
**assumes** *u*: *simple_function M u*
**assumes** *cong*: $\bigwedge f\ g.$ *simple_function M f* ⟹ *simple_function M g* ⟹ (*AE x in M*. $f\ x = g\ x$) ⟹ *P f* ⟹ *P g*
**assumes** *set*: $\bigwedge A.\ A \in$ *sets M* ⟹ *P* (*indicator A*)
**assumes** *mult*: $\bigwedge u\ c.$ *P u* ⟹ *P* ($\lambda x.\ c * u\ x$)
**assumes** *add*: $\bigwedge u\ v.$ *P u* ⟹ *P v* ⟹ *P* ($\lambda x.\ u\ x + v\ x$)
**and** *nn*: $\bigwedge x.\ u\ x \geq 0$
**shows** *P u*

**proposition** *simple_function_measurable_on_UNIV*:
**fixes** $f$ :: *'a::euclidean_space* ⇒ *real*
**assumes** *f*: *simple_function lebesgue f* **and** *nn*: $\bigwedge x.\ f\ x \geq 0$
**shows** *f measurable_on UNIV*

**corollary** *simple_function_measurable_on*:
**fixes** $f$ :: *'a::euclidean_space* ⇒ *real*
**assumes** *f*: *simple_function lebesgue f* **and** *nn*: $\bigwedge x.\ f\ x \geq 0$ **and** *S*: $S \in$ *sets lebesgue*
**shows** *f measurable_on S*

**proposition** *measurable_on_componentwise_UNIV*:
*f measurable_on UNIV* ⟷ (∀ *i*∈*Basis*. ($\lambda x.\ (f\ x \cdot i) *_R i$) *measurable_on UNIV*)
(**is** *?lhs* = *?rhs*)

**corollary** *measurable_on_componentwise*:
*f measurable_on S* ⟷ (∀ *i*∈*Basis*. ($\lambda x.\ (f\ x \cdot i) *_R i$) *measurable_on S*)

**lemma** *borel_measurable_implies_simple_function_sequence_real*:
**fixes** $u$ :: $'a \Rightarrow real$
**assumes** $u$[*measurable*]: $u \in$ *borel_measurable M* **and** *nn*: $\bigwedge x.\ u\ x \geq 0$
**shows** $\exists f.\ incseq\ f \wedge (\forall i.\ simple\_function\ M\ (f\ i)) \wedge (\forall x.\ bdd\_above\ (range\ (\lambda i.\ f\ i\ x))) \wedge$
$(\forall i\ x.\ 0 \leq f\ i\ x) \wedge u = (SUP\ i.\ f\ i)$

**proposition** *homeomorphic_box_UNIV*:
**fixes** $a\ b$:: $'a$::*euclidean_space*
**assumes** $box\ a\ b \neq \{\}$
**shows** *box a b homeomorphic* (*UNIV*::$'a$ *set*)

**proposition** *measurable_on_imp_borel_measurable_lebesgue_UNIV*:
**fixes** $f$ :: $'a$::*euclidean_space* $\Rightarrow$ $'b$::*euclidean_space*
**assumes** *f measurable_on UNIV*
**shows** $f \in$ *borel_measurable lebesgue*

**corollary** *measurable_on_imp_borel_measurable_lebesgue*:
**fixes** $f$ :: $'a$::*euclidean_space* $\Rightarrow$ $'b$::*euclidean_space*
**assumes** *f measurable_on S* **and** *S*: $S \in$ *sets lebesgue*
**shows** $f \in$ *borel_measurable* (*lebesgue_on S*)

**proposition** *measurable_on_limit*:
**fixes** $f$ :: *nat* $\Rightarrow$ $'a$::*euclidean_space* $\Rightarrow$ $'b$::*euclidean_space*
**assumes** *f*: $\bigwedge n.$ *f n measurable_on S* **and** *N*: *negligible N*
**and** *lim*: $\bigwedge x.\ x \in S - N \Longrightarrow (\lambda n.\ f\ n\ x) \longrightarrow g\ x$
**shows** *g measurable_on S*

**proposition** *lebesgue_measurable_imp_measurable_on*:
**fixes** $f$ :: $'a$::*euclidean_space* $\Rightarrow$ $'b$::*euclidean_space*
**assumes** *f*: $f \in$ *borel_measurable lebesgue* **and** *S*: $S \in$ *sets lebesgue*
**shows** *f measurable_on S*

**proposition** *measurable_on_iff_borel_measurable*:
**fixes** $f$ :: $'a$::*euclidean_space* $\Rightarrow$ $'b$::*euclidean_space*
**assumes** $S \in$ *sets lebesgue*
**shows** *f measurable_on S* $\longleftrightarrow$ $f \in$ *borel_measurable* (*lebesgue_on S*) (**is** *?lhs* = *?rhs*)

### 6.26.5 Monotonic functions are Lebesgue integrable

### 6.26.6 Measurability on generalisations of the binary product

**end**

## 6.27 Embedding Measure Spaces with a Function

**theory** *Embed_Measure*
**imports** *Binary_Product_Measure*
**begindefinition** *embed_measure* :: *'a measure* ⇒ (*'a* ⇒ *'b*) ⇒ *'b measure* **where**
*embed_measure M f* = *measure_of* (*f* ‘ *space M*) {*f* ‘ *A* |*A*. *A* ∈ *sets M*}
(λ*A*. *emeasure M* (*f* −‘ *A* ∩ *space M*))

**end**

## 6.28 Brouwer's Fixed Point Theorem

**theory** *Brouwer_Fixpoint*
**imports** *Homeomorphism Derivative*
**begin**

### 6.28.1 Retractions

### 6.28.2 Kuhn Simplices

### 6.28.3 Brouwer's fixed point theorem

**theorem** *brouwer*:
**fixes** *f* :: *'a::euclidean_space* ⇒ *'a*
**assumes** *S*: *compact S convex S S* ≠ {}
**and** *contf*: *continuous_on S f*
**and** *fim*: *f* ∈ *S* → *S*
**obtains** *x* **where** *x* ∈ *S* **and** *f x* = *x*

### 6.28.4 Applications

**corollary** *no_retraction_cball*:
**fixes** *a* :: *'a::euclidean_space*
**assumes** *e* > *0*
**shows** ¬ (*frontier* (*cball a e*) *retract_of* (*cball a e*))

**corollary** *contractible_sphere*:
**fixes** $a$ :: $'a$::*euclidean_space*
**shows** *contractible*(*sphere* $a$ $r$) $\longleftrightarrow r \leq 0$

**corollary** *connected_sphere_eq*:
**fixes** $a$ :: $'a$ :: *euclidean_space*
**shows** *connected*(*sphere* $a$ $r$) $\longleftrightarrow 2 \leq DIM('a) \vee r \leq 0$
(**is** *?lhs* = *?rhs*)

**corollary** *path_connected_sphere_eq*:
**fixes** $a$ :: $'a$ :: *euclidean_space*
**shows** *path_connected*(*sphere* $a$ $r$) $\longleftrightarrow 2 \leq DIM('a) \vee r \leq 0$
(**is** *?lhs* = *?rhs*)

**proposition** *frontier_subset_retraction*:
**fixes** $S$ :: $'a$::*euclidean_space set*
**assumes** *bounded* $S$ **and** *fros*: *frontier* $S \subseteq T$
**and** *contf*: *continuous_on* (*closure* $S$) $f$
**and** *fim*: $f \in S \to T$
**and** *fid*: $\bigwedge x.\ x \in T \Longrightarrow f\ x = x$
**shows** $S \subseteq T$

**corollary** *rel_frontier_retract_of_punctured_affine_hull*:
**fixes** $S$ :: $'a$::*euclidean_space set*
**assumes** *bounded* $S$ *convex* $S$ $a \in$ *rel_interior* $S$
**shows** *rel_frontier* $S$ *retract_of* (*affine hull* $S - \{a\}$)

**corollary** *rel_boundary_retract_of_punctured_affine_hull*:
**fixes** $S$ :: $'a$::*euclidean_space set*
**assumes** *compact* $S$ *convex* $S$ $a \in$ *rel_interior* $S$
**shows** ($S$ − *rel_interior* $S$) *retract_of* (*affine hull* $S - \{a\}$)
**theorem** *has_derivative_inverse_on*:
**fixes** $f$ :: $'n$::*euclidean_space* $\Rightarrow$ $'n$
**assumes** *open* $S$
**and** $\bigwedge x.\ x \in S \Longrightarrow$ ($f$ *has_derivative* $f'(x)$) (*at* $x$)
**and** $\bigwedge x.\ x \in S \Longrightarrow g\ (f\ x) = x$
**and** $f'\ x \circ g'\ x = id$
**and** $x \in S$
**shows** ($g$ *has_derivative* $g'(x)$) (*at* ($f\ x$))

**end**

## 6.29 Fashoda Meet Theorem

**theory** *Fashoda_Theorem*
**imports** *Brouwer_Fixpoint Path_Connected Cartesian_Euclidean_Space*
**begin**

### 6.29.1 Bijections between intervals

**definition** *interval_bij* :: *'a × 'a ⇒ 'a × 'a ⇒ 'a ⇒ 'a::euclidean_space*
  **where** *interval_bij* =
    (λ(*a*, *b*) (*u*, *v*) *x*. (∑ *i*∈*Basis*. (*u*·*i* + (*x*·*i* − *a*·*i*) / (*b*·*i* − *a*·*i*) ∗ (*v*·*i* − *u*·*i*)) $*_R$ *i*))

### 6.29.2 Fashoda meet theorem

**proposition** *fashoda_unit*:
  **fixes** *f g* :: *real ⇒ real^2*
  **assumes** *f* ‘ {−*1* .. *1*} ⊆ *cbox* (−*1*) *1*
    **and** *g* ‘ {−*1* .. *1*} ⊆ *cbox* (−*1*) *1*
    **and** *continuous_on* {−*1* .. *1*} *f*
    **and** *continuous_on* {−*1* .. *1*} *g*
    **and** *f* (− *1*)$*1* = − *1*
    **and** *f* *1*$*1* = *1* *g* (− *1*) $*2* = −*1*
    **and** *g* *1* $*2* = *1*
  **shows** ∃ *s*∈{−*1* .. *1*}. ∃ *t*∈{−*1* .. *1*}. *f s* = *g t*

**proposition** *fashoda_unit_path*:
  **fixes** *f g* :: *real ⇒ real^2*
  **assumes** *path f*
    **and** *path g*
    **and** *path_image f* ⊆ *cbox* (−*1*) *1*
    **and** *path_image g* ⊆ *cbox* (−*1*) *1*
    **and** (*pathstart f*)$*1* = −*1*
    **and** (*pathfinish f*)$*1* = *1*
    **and** (*pathstart g*)$*2* = −*1*
    **and** (*pathfinish g*)$*2* = *1*
  **obtains** *z* **where** *z* ∈ *path_image f* **and** *z* ∈ *path_image g*

**theorem** *fashoda*:
  **fixes** *b* :: *real^2*
  **assumes** *path f*
    **and** *path g*
    **and** *path_image f* ⊆ *cbox a b*
    **and** *path_image g* ⊆ *cbox a b*
    **and** (*pathstart f*)$*1* = *a*$*1*
    **and** (*pathfinish f*)$*1* = *b*$*1*
    **and** (*pathstart g*)$*2* = *a*$*2*
    **and** (*pathfinish g*)$*2* = *b*$*2*
  **obtains** *z* **where** *z* ∈ *path_image f* **and** *z* ∈ *path_image g*

### 6.29.3 Useful Fashoda corollary pointed out to me by Tom Hales

**corollary** *fashoda_interlace*:
**fixes** $a$ :: *real^2*
**assumes** *path f*
**and** *path g*
**and** *paf*: *path_image f* ⊆ *cbox a b*
**and** *pag*: *path_image g* ⊆ *cbox a b*
**and** (*pathstart f*)$2 = a$2
**and** (*pathfinish f*)$2 = a$2
**and** (*pathstart g*)$2 = a$2
**and** (*pathfinish g*)$2 = a$2
**and** (*pathstart f*)$1 < (*pathstart g*)$1
**and** (*pathstart g*)$1 < (*pathfinish f*)$1
**and** (*pathfinish f*)$1 < (*pathfinish g*)$1
**obtains** *z* **where** *z* ∈ *path_image f* **and** *z* ∈ *path_image g*

**end**

## 6.30 Vector Cross Products in 3 Dimensions

**theory** *Cross3*
**imports** *Determinants Cartesian_Euclidean_Space*
**begin**

**definition** *cross3* :: [*real^3*, *real^3*] ⇒ *real^3* (**infixr** × *80*)
**where** *a* × *b* ≡
*vector* [a$2 * b$3 − a$3 * b$2,
a$3 * b$1 − a$1 * b$3,
a$1 * b$2 − a$2 * b$1]

### 6.30.1 Basic lemmas

**proposition** *Jacobi*: $x \times (y \times z) + y \times (z \times x) + z \times (x \times y) = 0$ **for** *x*::*real^3*

**proposition** *Lagrange*: $x \times (y \times z) = (x \cdot z) *_R y - (x \cdot y) *_R z$

**proposition** *cross_triple*: $(x \times y) \cdot z = (y \times z) \cdot x$

**proposition** *dot_cross*: $(w \times x) \cdot (y \times z) = (w \cdot y) * (x \cdot z) - (w \cdot z) * (x \cdot y)$

**proposition** *norm_cross*: $(norm\ (x \times y))^2 = (norm\ x)^2 * (norm\ y)^2 - (x \cdot y)^2$

### 6.30.2 Preservation by rotation, or other orthogonal transformation up to sign

### 6.30.3 Continuity

**end**

## 6.31 Bounded Continuous Functions

**theory** *Bounded_Continuous_Function*
**imports**
*Topology_Euclidean_Space*
*Uniform_Limit*
**begin**

### 6.31.1 Definition

**definition** *bcontfun* = $\{f.$ *continuous_on UNIV f* $\wedge$ *bounded* (*range f*)$\}$

**instantiation** *bcontfun* :: (*topological_space*, *metric_space*) *metric_space*
**begin**

**lift_definition** *dist_bcontfun* :: $'a \Rightarrow_C 'b \Rightarrow 'a \Rightarrow_C 'b \Rightarrow$ *real*
**is** $\lambda f\ g.$ (*SUP x. dist* (*f x*) (*g x*))

### 6.31.2 Complete Space

**instance** *bcontfun* :: (*metric_space*, *complete_space*) *complete_space*

**end**

## 6.32 Infinite Products

**theory** *Infinite_Products*
**imports** *Topology_Euclidean_Space Complex_Transcendental*
**begin**

### 6.32.1 Definitions and basic properties

**definition** *raw_has_prod* :: [*nat* $\Rightarrow$ $'a$::{*t2_space*, *comm_semiring_1*}, *nat*, $'a$] $\Rightarrow$ *bool*
**where** *raw_has_prod f M p* $\equiv$ $(\lambda n. \prod i{\le}n.\ f\ (i{+}M)) \longrightarrow p \wedge p \neq 0$

**definition**
*has_prod* :: (*nat* $\Rightarrow$ $'a$::{*t2_space*, *comm_semiring_1*}) $\Rightarrow$ $'a \Rightarrow$ *bool* (**infixr** *has′_prod 80*)

**where** *f has_prod p* ≡ *raw_has_prod f 0 p* ∨ (∃ *i q*. *p* = *0* ∧ *f i* = *0* ∧ *raw_has_prod f* (*Suc i*) *q*)

**definition** *convergent_prod* :: (*nat* ⇒ *'a* :: {*t2_space*,*comm_semiring_1*}) ⇒ *bool* **where**
*convergent_prod f* ≡ ∃ *M p*. *raw_has_prod f M p*

**definition** *prodinf* :: (*nat* ⇒ *'a*::{*t2_space*, *comm_semiring_1*}) ⇒ *'a*
(**binder** ∏ *10*)
**where** *prodinf f* = (*THE p*. *f has_prod p*)

### 6.32.2 Absolutely convergent products

**definition** *abs_convergent_prod* :: (*nat* ⇒ _) ⇒ *bool* **where**
*abs_convergent_prod f* ⟷ *convergent_prod* (λ*i*. *1* + *norm* (*f i* − *1*))

**lemma** *convergent_prod_iff_convergent*:
**fixes** *f* :: *nat* ⇒ *'a* :: {*topological_semigroup_mult*,*t2_space*,*idom*}
**assumes** ⋀*i*. *f i* ≠ *0*
**shows** *convergent_prod f* ⟷ *convergent* (λ*n*. ∏ *i*≤*n*. *f i*) ∧ *lim* (λ*n*. ∏ *i*≤*n*. *f i*) ≠ *0*

**theorem** *abs_convergent_prod_conv_summable*:
**fixes** *f* :: *nat* ⇒ *'a* :: *real_normed_div_algebra*
**shows** *abs_convergent_prod f* ⟷ *summable* (λ*i*. *norm* (*f i* − *1*))

### 6.32.3 More elementary properties

**theorem** *abs_convergent_prod_imp_convergent_prod*:
**fixes** *f* :: *nat* ⇒ *'a* :: {*real_normed_div_algebra*,*complete_space*,*comm_ring_1*}
**assumes** *abs_convergent_prod f*
**shows** *convergent_prod f*

**corollary** *convergent_prod_offset_0*:
**fixes** *f* :: *nat* ⇒ *'a* :: {*idom*,*topological_semigroup_mult*,*t2_space*}
**assumes** *convergent_prod f* ⋀*i*. *f i* ≠ *0*
**shows** ∃ *p*. *raw_has_prod f 0 p*

**theorem** *has_prod_iff*: *f has_prod x* ⟷ *convergent_prod f* ∧ *prodinf f* = *x*

### 6.32.4 Exponentials and logarithms

**theorem** *convergent_prod_iff_summable_real*:
**fixes** *a* :: *nat* ⇒ *real*

**assumes** $\bigwedge n.\ a\ n > 0$
**shows** *convergent_prod* (λk. *1* + *a k*) ⟷ *summable a* (**is** *?lhs* = *?rhs*)

**theorem** *Ln_prodinf_complex*:
**fixes** *z* :: *nat* ⇒ *complex*
**assumes** *z*: $\bigwedge j.\ z\ j \neq 0$ **and** ξ: $\xi \neq 0$
**shows** ((λn. $\prod j \leq n.\ z\ j$) ⟶ ξ) ⟷ (∃ k. (λn. ($\sum j \leq n.\ Ln\ (z\ j)$)) ⟶ *Ln* ξ + *of_int k* ∗ (*of_real*(*2*∗*pi*) ∗ i)) (**is** *?lhs* = *?rhs*)
**proposition** *convergent_prod_iff_summable_complex*:
**fixes** *z* :: *nat* ⇒ *complex*
**assumes** $\bigwedge k.\ z\ k \neq 0$
**shows** *convergent_prod* (λk. *z k*) ⟷ *summable* (λk. *Ln* (*z k*)) (**is** *?lhs* = *?rhs*)
**proposition** *summable_imp_convergent_prod_complex*:
**fixes** *z* :: *nat* ⇒ *complex*
**assumes** *z*: *summable* (λk. *norm* (*z k*)) **and** *non0*: $\bigwedge k.\ z\ k \neq -1$
**shows** *convergent_prod* (λk. *1* + *z k*)

**corollary** *summable_imp_convergent_prod_real*:
**fixes** *z* :: *nat* ⇒ *real*
**assumes** *z*: *summable* (λk. |*z k*|) **and** *non0*: $\bigwedge k.\ z\ k \neq -1$
**shows** *convergent_prod* (λk. *1* + *z k*)

**end**

## 6.33 Sums over Infinite Sets

**theory** *Infinite_Set_Sum*
**imports** *Set_Integral Infinite_Sum*
**begin**

**definition** *abs_summable_on* ::
('*a* ⇒ '*b* :: {*banach*, *second_countable_topology*}) ⇒ '*a set* ⇒ *bool*
(**infix** *abs'_summable'_on 50*)
**where**
*f abs_summable_on A* ⟷ *integrable* (*count_space A*) *f*

**definition** *infsetsum* ::
('*a* ⇒ '*b* :: {*banach*, *second_countable_topology*}) ⇒ '*a set* ⇒ '*b*
**where**
*infsetsum f A* = *lebesgue_integral* (*count_space A*) *f*

**theorem** *infsetsum_reindex*:
**assumes** *inj_on g A*

**shows** *infsetsum f* $(g$ ‘ $A) =$ *infsetsum* $(\lambda x.\ f\ (g\ x))\ A$

**theorem** *infsetsum_Sigma*:
**fixes** $A$ :: *'a set* **and** $B$ :: *'a* ⇒ *'b set*
**assumes** [*simp*]: *countable* $A$ **and** $\bigwedge i.$ *countable* $(B\ i)$
**assumes** *summable*: *f abs_summable_on* (*Sigma* $A$ $B$)
**shows** *infsetsum f* (*Sigma* $A$ $B$) = *infsetsum* ($\lambda x.$ *infsetsum* ($\lambda y.\ f\ (x, y)$) ($B$ $x$)) $A$

**theorem** *abs_summable_on_Sigma_iff*:
**assumes** [*simp*]: *countable* $A$ **and** $\bigwedge x.\ x \in A \Longrightarrow$ *countable* $(B\ x)$
**shows** *f abs_summable_on Sigma A B* ⟷
($\forall x \in A.$ ($\lambda y.\ f\ (x, y)$) *abs_summable_on* $B\ x$) ∧
(($\lambda x.$ *infsetsum* ($\lambda y.$ *norm* ($f\ (x, y)$)) ($B\ x$)) *abs_summable_on* $A$)

**theorem** *infsetsum_prod_PiE*:
**fixes** $f$ :: *'a* ⇒ *'b* ⇒ *'c* :: {*real_normed_field*,*banach*,*second_countable_topology*}
**assumes** *finite*: *finite* $A$ **and** *countable*: $\bigwedge x.\ x \in A \Longrightarrow$ *countable* $(B\ x)$
**assumes** *summable*: $\bigwedge x.\ x \in A \Longrightarrow$ *f x abs_summable_on B x*
**shows** *infsetsum* ($\lambda g.\ \prod x \in A.\ f\ x\ (g\ x)$) (*PiE* $A$ $B$) = ($\prod x \in A.$ *infsetsum* $(f\ x)$ $(B\ x)$)

**end**

## 6.34 Faces, Extreme Points, Polytopes, Polyhedra etc

**theory** *Polytope*
**imports** *Cartesian_Euclidean_Space Path_Connected*
**begin**

### 6.34.1 Faces of a (usually convex) set

**definition** *face_of* :: [*'a*::*real_vector set*, *'a set*] ⇒ *bool* (**infixr** (*face'_of*) *50*)
**where**
*T face_of S* ⟷
$T \subseteq S \wedge$ *convex* $T$ ∧
($\forall a \in S.\ \forall b \in S.\ \forall x \in T.\ x \in$ *open_segment* $a\ b \longrightarrow a \in T \wedge b \in T$)

**proposition** *face_of_imp_eq_affine_Int*:
**fixes** $S$ :: *'a*::*euclidean_space set*
**assumes** $S$: *convex* $S$ **and** $T$: *T face_of S*
**shows** $T =$ (*affine hull* $T$) $\cap S$

**proposition** *face_of_convex_hulls*:
**assumes** $S$: *finite* $S$ $T \subseteq S$ **and** *disj*: *affine hull* $T \cap$ *convex hull* $(S - T) = \{\}$
**shows** (*convex hull* $T$) *face_of* (*convex hull* $S$)

**proposition** *face_of_convex_hull_insert*:
**assumes** *finite* $S$ $a \notin$ *affine hull* $S$ **and** $T$: $T$ *face_of convex hull* $S$
**shows** $T$ *face_of convex hull insert* $a$ $S$

**proposition** *face_of_affine_trivial*:
**assumes** *affine* $S$ $T$ *face_of* $S$
**shows** $T = \{\} \vee T = S$

**proposition** *Inter_faces_finite_altbound*:
**fixes** $T$ :: *'a::euclidean_space set set*
**assumes** *cfaI*: $\bigwedge c.\ c \in T \Longrightarrow c$ *face_of* $S$
**shows** $\exists F'.$ *finite* $F' \wedge F' \subseteq T \wedge$ *card* $F' \leq DIM('a) + 2 \wedge \bigcap F' = \bigcap T$

**proposition** *face_of_Times*:
**assumes** $F$ *face_of* $S$ **and** $F'$ *face_of* $S'$
**shows** $(F \times F')$ *face_of* $(S \times S')$

**corollary** *face_of_Times_decomp*:
**fixes** $S$ :: *'a::euclidean_space set* **and** $S'$ :: *'b::euclidean_space set*
**shows** $C$ *face_of* $(S \times S') \longleftrightarrow (\exists F\ F'.\ F$ *face_of* $S \wedge F'$ *face_of* $S' \wedge C = F \times F')$
(**is** *?lhs* = *?rhs*)

### 6.34.2 Exposed faces

**definition** *exposed_face_of* :: [*'a::euclidean_space set*, *'a set*] ⇒ *bool*
(**infixr** (*exposed'_face'_of*) *50*)
**where** $T$ *exposed_face_of* $S \longleftrightarrow$
$T$ *face_of* $S \wedge (\exists a\ b.\ S \subseteq \{x.\ a \cdot x \leq b\} \wedge T = S \cap \{x.\ a \cdot x = b\})$

**proposition** *exposed_face_of_Int*:
**assumes** $T$ *exposed_face_of* $S$
**and** $U$ *exposed_face_of* $S$
**shows** $(T \cap U)$ *exposed_face_of* $S$

**proposition** *exposed_face_of_Inter*:
**fixes** $P$ :: *'a::euclidean_space set set*
**assumes** $P \neq \{\}$
**and** $\bigwedge T.\ T \in P \Longrightarrow T$ *exposed_face_of* $S$
**shows** $\bigcap P$ *exposed_face_of* $S$

**proposition** *exposed_face_of_sums*:
**assumes** *convex S* **and** *convex T*
**and** *F exposed_face_of* $\{x + y \mid x\ y.\ x \in S \wedge y \in T\}$
(**is** *F exposed_face_of ?ST*)
**obtains** *k l*
**where** *k exposed_face_of S l exposed_face_of T*
$F = \{x + y \mid x\ y.\ x \in k \wedge y \in l\}$

**proposition** *exposed_face_of_parallel*:
*T exposed_face_of S* $\longleftrightarrow$
*T face_of S* $\wedge$
$(\exists a\ b.\ S \subseteq \{x.\ a \cdot x \le b\} \wedge T = S \cap \{x.\ a \cdot x = b\} \wedge$
$(T \neq \{\} \longrightarrow T \neq S \longrightarrow a \neq 0) \wedge$
$(T \neq S \longrightarrow (\forall w \in \mathit{affine\ hull}\ S.\ (w + a) \in \mathit{affine\ hull}\ S)))$
(**is** *?lhs = ?rhs*)

### 6.34.3 Extreme points of a set: its singleton faces

**definition** *extreme_point_of* :: *['a::real_vector, 'a set] ⇒ bool*
(**infixr** (*extreme'_point'_of*) *50*)
**where** *x extreme_point_of S* $\longleftrightarrow$
$x \in S \wedge (\forall a \in S.\ \forall b \in S.\ x \notin \mathit{open\_segment}\ a\ b)$

**proposition** *extreme_points_of_convex_hull*:
$\{x.\ x$ *extreme_point_of* (*convex hull S*)$\} \subseteq S$

### 6.34.4 Facets

**definition** *facet_of* :: *['a::euclidean_space set, 'a set] ⇒ bool*
(**infixr** (*facet'_of*) *50*)
**where** *F facet_of S* $\longleftrightarrow$ *F face_of S* $\wedge F \neq \{\} \wedge$ *aff_dim F = aff_dim S − 1*

### 6.34.5 Edges: faces of affine dimension 1

**definition** *edge_of* :: *['a::euclidean_space set, 'a set] ⇒ bool* (**infixr** (*edge'_of*) *50*)
**where** *e edge_of S* $\longleftrightarrow$ *e face_of S* $\wedge$ *aff_dim e = 1*

### 6.34.6 Existence of extreme points

**proposition** *different_norm_3_collinear_points*:
**fixes** *a* :: *'a::euclidean_space*
**assumes** $x \in$ *open_segment a b norm(a) = norm(b) norm(x) = norm(b)*
**shows** *False*

**proposition** *extreme_point_exists_convex*:

**fixes** $S$ :: $'a$::*euclidean_space set*
**assumes** *compact* $S$ *convex* $S$ $S \neq \{\}$
**obtains** $x$ **where** $x$ *extreme_point_of* $S$

### 6.34.7 Krein-Milman, the weaker form

**proposition** *Krein_Milman*:
**fixes** $S$ :: $'a$::*euclidean_space set*
**assumes** *compact* $S$ *convex* $S$
**shows** $S$ = *closure*(*convex hull* $\{x.\ x$ *extreme_point_of* $S\}$)

**theorem** *Krein_Milman_Minkowski*:
**fixes** $S$ :: $'a$::*euclidean_space set*
**assumes** *compact* $S$ *convex* $S$
**shows** $S$ = *convex hull* $\{x.\ x$ *extreme_point_of* $S\}$

### 6.34.8 Applying it to convex hulls of explicitly indicated finite sets

**corollary** *Krein_Milman_polytope*:
**fixes** $S$ :: $'a$::*euclidean_space set*
**shows**
*finite* $S$
$\Longrightarrow$ *convex hull* $S$ =
*convex hull* $\{x.\ x$ *extreme_point_of* (*convex hull* $S$)$\}$

**proposition** *face_of_convex_hull_insert_eq*:
**fixes** $a$ :: $'a$ :: *euclidean_space*
**assumes** *finite* $S$ **and** $a$: $a \notin$ *affine hull* $S$
**shows** ($F$ *face_of* (*convex hull* (*insert* $a$ $S$)) $\longleftrightarrow$
$F$ *face_of* (*convex hull* $S$) $\vee$
($\exists F'.\ F'$ *face_of* (*convex hull* $S$) $\wedge$ $F$ = *convex hull* (*insert* $a$ $F'$)))
(**is** $F$ *face_of* *?CAS* $\longleftrightarrow$ _)

**proposition** *face_of_convex_hull_affine_independent*:
**fixes** $S$ :: $'a$::*euclidean_space set*
**assumes** $\neg$ *affine_dependent* $S$
**shows** ($T$ *face_of* (*convex hull* $S$) $\longleftrightarrow$ ($\exists c.\ c \subseteq S \wedge T$ = *convex hull* $c$))
(**is** *?lhs* = *?rhs*)

**proposition** *Krein_Milman_frontier*:
**fixes** $S$ :: $'a$::*euclidean_space set*
**assumes** *convex* $S$ *compact* $S$

**shows** $S = convex\ hull\ (frontier\ S)$
(**is** $?lhs = ?rhs$)

### 6.34.9 Polytopes

**definition** *polytope* **where**
$polytope\ S \equiv \exists\, v.\ finite\ v \wedge S = convex\ hull\ v$

**proposition** *face_of_polytope_insert2*:
**fixes** $a :: 'a :: euclidean\_space$
**assumes** $polytope\ S\ a \notin affine\ hull\ S\ F\ face\_of\ S$
**shows** $convex\ hull\ (insert\ a\ F)\ face\_of\ convex\ hull\ (insert\ a\ S)$

### 6.34.10 Polyhedra

**definition** *polyhedron* **where**
$polyhedron\ S \equiv$
$\exists\, F.\ finite\ F \wedge$
$S = \bigcap\ F \wedge$
$(\forall\, h \in F.\ \exists\, a\ b.\ a \neq 0 \wedge h = \{x.\ a \cdot x \leq b\})$

### 6.34.11 Canonical polyhedron representation making facial structure explicit

**proposition** *polyhedron_Int_affine*:
**fixes** $S :: 'a :: euclidean\_space\ set$
**shows** $polyhedron\ S \longleftrightarrow$
$(\exists\, F.\ finite\ F \wedge S = (affine\ hull\ S) \cap \bigcap F \wedge$
$(\forall\, h \in F.\ \exists\, a\ b.\ a \neq 0 \wedge h = \{x.\ a \cdot x \leq b\}))$

**proposition** *rel_interior_polyhedron_explicit*:
**assumes** $finite\ F$
**and** *seq*: $S = affine\ hull\ S \cap \bigcap F$
**and** *faceq*: $\bigwedge h.\ h \in F \Longrightarrow a\ h \neq 0 \wedge h = \{x.\ a\ h \cdot x \leq b\ h\}$
**and** *psub*: $\bigwedge F'.\ F' \subset F \Longrightarrow S \subset affine\ hull\ S \cap \bigcap F'$
**shows** $rel\_interior\ S = \{x \in S.\ \forall\, h \in F.\ a\ h \cdot x < b\ h\}$

**proposition** *polyhedron_Int_affine_parallel_minimal*:
**fixes** $S :: 'a :: euclidean\_space\ set$
**shows** $polyhedron\ S \longleftrightarrow$
$(\exists\, F.\ finite\ F \wedge$
$S = (affine\ hull\ S) \cap (\bigcap F) \wedge$
$(\forall\, h \in F.\ \exists\, a\ b.\ a \neq 0 \wedge h = \{x.\ a \cdot x \leq b\} \wedge$
$(\forall\, x \in affine\ hull\ S.\ (x + a) \in affine\ hull\ S)) \wedge$

$(\forall F'. F' \subset F \longrightarrow S \subset (\mathit{affine\ hull}\ S) \cap (\bigcap F')))$
(**is** $?lhs = ?rhs$)

**proposition** *facet_of_polyhedron_explicit*:
**assumes** *finite* $F$
**and** *seq*: $S = \mathit{affine\ hull}\ S \cap \bigcap F$
**and** *faceq*: $\bigwedge h.\ h \in F \Longrightarrow a\ h \neq 0 \wedge h = \{x.\ a\ h \cdot x \leq b\ h\}$
**and** *psub*: $\bigwedge F'.\ F' \subset F \Longrightarrow S \subset \mathit{affine\ hull}\ S \cap \bigcap F'$
**shows** $C\ \mathit{facet\_of}\ S \longleftrightarrow (\exists h.\ h \in F \wedge C = S \cap \{x.\ a\ h \cdot x = b\ h\})$

**proposition** *face_of_polyhedron_explicit*:
**fixes** $S :: 'a :: \mathit{euclidean\_space\ set}$
**assumes** *finite* $F$
**and** *seq*: $S = \mathit{affine\ hull}\ S \cap \bigcap F$
**and** *faceq*: $\bigwedge h.\ h \in F \Longrightarrow a\ h \neq 0 \wedge h = \{x.\ a\ h \cdot x \leq b\ h\}$
**and** *psub*: $\bigwedge F'.\ F' \subset F \Longrightarrow S \subset \mathit{affine\ hull}\ S \cap \bigcap F'$
**and** $C$: $C\ \mathit{face\_of}\ S$ **and** $C \neq \{\}$ $C \neq S$
**shows** $C = \bigcap\{S \cap \{x.\ a\ h \cdot x = b\ h\} \mid h.\ h \in F \wedge C \subseteq S \cap \{x.\ a\ h \cdot x = b\ h\}\}$

### 6.34.12 More general corollaries from the explicit representation

**corollary** *facet_of_polyhedron*:
**assumes** *polyhedron* $S$ **and** $C\ \mathit{facet\_of}\ S$
**obtains** $a\ b$ **where** $a \neq 0$ $S \subseteq \{x.\ a \cdot x \leq b\}$ $C = S \cap \{x.\ a \cdot x = b\}$

**corollary** *face_of_polyhedron*:
**assumes** *polyhedron* $S$ **and** $C\ \mathit{face\_of}\ S$ **and** $C \neq \{\}$ **and** $C \neq S$
**shows** $C = \bigcap\{F.\ F\ \mathit{facet\_of}\ S \wedge C \subseteq F\}$

**proposition** *rel_interior_of_polyhedron*:
**fixes** $S :: 'a :: \mathit{euclidean\_space\ set}$
**assumes** *polyhedron* $S$
**shows** $\mathit{rel\_interior}\ S = S - \bigcup\{F.\ F\ \mathit{facet\_of}\ S\}$

**proposition** *polyhedron_eq_finite_exposed_faces*:
**fixes** $S :: 'a :: \mathit{euclidean\_space\ set}$
**shows** $\mathit{polyhedron}\ S \longleftrightarrow \mathit{closed}\ S \wedge \mathit{convex}\ S \wedge \mathit{finite}\ \{F.\ F\ \mathit{exposed\_face\_of}\ S\}$
(**is** $?lhs = ?rhs$)

**corollary** *polyhedron_eq_finite_faces*:
**fixes** $S :: 'a :: \mathit{euclidean\_space\ set}$
**shows** $\mathit{polyhedron}\ S \longleftrightarrow \mathit{closed}\ S \wedge \mathit{convex}\ S \wedge \mathit{finite}\ \{F.\ F\ \mathit{face\_of}\ S\}$
(**is** $?lhs = ?rhs$)

### 6.34.13 Relation between polytopes and polyhedra

**proposition** *polytope_eq_bounded_polyhedron*:
**fixes** $S$ :: $'a$ :: *euclidean_space set*
**shows** *polytope* $S \longleftrightarrow$ *polyhedron* $S \wedge$ *bounded* $S$
(**is** *?lhs* = *?rhs*)

### 6.34.14 Relative and absolute frontier of a polytope

**proposition** *frontier_of_convex_hull*:
**fixes** $S$ :: $'a$::*euclidean_space set*
**assumes** *card* $S$ = *Suc* (*DIM*($'a$))
**shows** *frontier*(*convex hull* $S$) = $\bigcup$ {*convex hull* ($S$ − {$a$}) | $a$. $a \in S$}

### 6.34.15 Special case of a triangle

**proposition** *frontier_of_triangle*:
**fixes** $a$ :: $'a$::*euclidean_space*
**assumes** *DIM*($'a$) = *2*
**shows** *frontier*(*convex hull* {$a$,$b$,$c$}) = *closed_segment* $a$ $b$ ∪ *closed_segment* $b$ $c$ ∪ *closed_segment* $c$ $a$
(**is** *?lhs* = *?rhs*)

**corollary** *inside_of_triangle*:
**fixes** $a$ :: $'a$::*euclidean_space*
**assumes** *DIM*($'a$) = *2*
**shows** *inside* (*closed_segment* $a$ $b$ ∪ *closed_segment* $b$ $c$ ∪ *closed_segment* $c$ $a$) = *interior*(*convex hull* {$a$,$b$,$c$})

**corollary** *interior_of_triangle*:
**fixes** $a$ :: $'a$::*euclidean_space*
**assumes** *DIM*($'a$) = *2*
**shows** *interior*(*convex hull* {$a$,$b$,$c$}) =
*convex hull* {$a$,$b$,$c$} − (*closed_segment* $a$ $b$ ∪ *closed_segment* $b$ $c$ ∪ *closed_segment* $c$ $a$)

### 6.34.16 Subdividing a cell complex

**proposition** *cell_complex_subdivision_exists*:
**fixes** $\mathcal{F}$ :: $'a$::*euclidean_space set set*
**assumes** $0 < e$ *finite* $\mathcal{F}$
**and** *poly*: $\bigwedge X.\ X \in \mathcal{F} \Longrightarrow$ *polytope* $X$
**and** *aff*: $\bigwedge X.\ X \in \mathcal{F} \Longrightarrow$ *aff_dim* $X \le d$

**and** *face*: $\bigwedge X\ Y.\ [\![X \in \mathcal{F};\ Y \in \mathcal{F}]\!] \Longrightarrow X \cap Y\ \mathit{face\_of}\ X$
**obtains** $\mathcal{F}'$ **where** $\mathit{finite}\ \mathcal{F}'\ \bigcup\mathcal{F}' = \bigcup\mathcal{F}\ \bigwedge X.\ X \in \mathcal{F}' \Longrightarrow \mathit{diameter}\ X < e$
$\bigwedge X.\ X \in \mathcal{F}' \Longrightarrow \mathit{polytope}\ X\ \bigwedge X.\ X \in \mathcal{F}' \Longrightarrow \mathit{aff\_dim}\ X \le d$
$\bigwedge X\ Y.\ [\![X \in \mathcal{F}';\ Y \in \mathcal{F}']\!] \Longrightarrow X \cap Y\ \mathit{face\_of}\ X$
$\bigwedge C.\ C \in \mathcal{F}' \Longrightarrow \exists D.\ D \in \mathcal{F} \wedge C \subseteq D$
$\bigwedge C\ x.\ C \in \mathcal{F} \wedge x \in C \Longrightarrow \exists D.\ D \in \mathcal{F}' \wedge x \in D \wedge D \subseteq C$

### 6.34.17 Simplexes

**definition** *simplex* :: *int* ⇒ *'a::euclidean_space set* ⇒ *bool* (**infix** *simplex 50*)
**where** $n\ \mathit{simplex}\ S \equiv \exists C.\ \neg\ \mathit{affine\_dependent}\ C \wedge \mathit{int}(\mathit{card}\ C) = n + 1 \wedge S = \mathit{convex\ hull}\ C$

### 6.34.18 Simplicial complexes and triangulations

**definition** *simplicial_complex* **where**
$\mathit{simplicial\_complex}\ \mathcal{C} \equiv$
$\mathit{finite}\ \mathcal{C} \wedge$
$(\forall S \in \mathcal{C}.\ \exists n.\ n\ \mathit{simplex}\ S) \wedge$
$(\forall F\ S.\ S \in \mathcal{C} \wedge F\ \mathit{face\_of}\ S \longrightarrow F \in \mathcal{C}) \wedge$
$(\forall S\ S'.\ S \in \mathcal{C} \wedge S' \in \mathcal{C} \longrightarrow (S \cap S')\ \mathit{face\_of}\ S)$

**definition** *triangulation* **where**
$\mathit{triangulation}\ \mathcal{T} \equiv$
$\mathit{finite}\ \mathcal{T} \wedge$
$(\forall T \in \mathcal{T}.\ \exists n.\ n\ \mathit{simplex}\ T) \wedge$
$(\forall T\ T'.\ T \in \mathcal{T} \wedge T' \in \mathcal{T} \longrightarrow (T \cap T')\ \mathit{face\_of}\ T)$

### 6.34.19 Refining a cell complex to a simplicial complex

**proposition** *convex_hull_insert_Int_eq*:
**fixes** *z* :: *'a* :: *euclidean_space*
**assumes** $z$: $z \in \mathit{rel\_interior}\ S$
**and** $T$: $T \subseteq \mathit{rel\_frontier}\ S$
**and** $U$: $U \subseteq \mathit{rel\_frontier}\ S$
**and** *convex S convex T convex U*
**shows** $\mathit{convex\ hull}\ (\mathit{insert}\ z\ T) \cap \mathit{convex\ hull}\ (\mathit{insert}\ z\ U) = \mathit{convex\ hull}\ (\mathit{insert}\ z\ (T \cap U))$
(**is** *?lhs* = *?rhs*)

**proposition** *simplicial_subdivision_of_cell_complex*:
**assumes** $\mathit{finite}\ \mathcal{M}$
**and** *poly*: $\bigwedge C.\ C \in \mathcal{M} \Longrightarrow \mathit{polytope}\ C$
**and** *face*: $\bigwedge C1\ C2.\ [\![C1 \in \mathcal{M};\ C2 \in \mathcal{M}]\!] \Longrightarrow C1 \cap C2\ \mathit{face\_of}\ C1$
**obtains** $\mathcal{T}$ **where** $\mathit{simplicial\_complex}\ \mathcal{T}$
$\bigcup\mathcal{T} = \bigcup\mathcal{M}$

$\bigwedge C.\ C \in \mathcal{M} \Longrightarrow \exists F.\ \mathit{finite}\ F \wedge F \subseteq \mathcal{T} \wedge C = \bigcup F$
$\bigwedge K.\ K \in \mathcal{T} \Longrightarrow \exists C.\ C \in \mathcal{M} \wedge K \subseteq C$

**corollary** *fine_simplicial_subdivision_of_cell_complex*:
**assumes** $0 < e$ *finite* $\mathcal{M}$
**and** *poly*: $\bigwedge C.\ C \in \mathcal{M} \Longrightarrow \mathit{polytope}\ C$
**and** *face*: $\bigwedge C1\ C2.\ [\![C1 \in \mathcal{M};\ C2 \in \mathcal{M}]\!] \Longrightarrow C1 \cap C2\ \mathit{face\_of}\ C1$
**obtains** $\mathcal{T}$ **where** *simplicial_complex* $\mathcal{T}$
$\bigwedge K.\ K \in \mathcal{T} \Longrightarrow \mathit{diameter}\ K < e$
$\bigcup \mathcal{T} = \bigcup \mathcal{M}$
$\bigwedge C.\ C \in \mathcal{M} \Longrightarrow \exists F.\ \mathit{finite}\ F \wedge F \subseteq \mathcal{T} \wedge C = \bigcup F$
$\bigwedge K.\ K \in \mathcal{T} \Longrightarrow \exists C.\ C \in \mathcal{M} \wedge K \subseteq C$

### 6.34.20 Some results on cell division with full-dimensional cells only

**proposition** *fine_triangular_subdivision_of_cell_complex*:
**assumes** $0 < e$ *finite* $\mathcal{M}$
**and** *poly*: $\bigwedge C.\ C \in \mathcal{M} \Longrightarrow \mathit{polytope}\ C$
**and** *aff*: $\bigwedge C.\ C \in \mathcal{M} \Longrightarrow \mathit{aff\_dim}\ C = d$
**and** *face*: $\bigwedge C1\ C2.\ [\![C1 \in \mathcal{M};\ C2 \in \mathcal{M}]\!] \Longrightarrow C1 \cap C2\ \mathit{face\_of}\ C1$
**obtains** $\mathcal{T}$ **where** *triangulation* $\mathcal{T}$ $\bigwedge k.\ k \in \mathcal{T} \Longrightarrow \mathit{diameter}\ k < e$
$\bigwedge k.\ k \in \mathcal{T} \Longrightarrow \mathit{aff\_dim}\ k = d$ $\bigcup \mathcal{T} = \bigcup \mathcal{M}$
$\bigwedge C.\ C \in \mathcal{M} \Longrightarrow \exists f.\ \mathit{finite}\ f \wedge f \subseteq \mathcal{T} \wedge C = \bigcup f$
$\bigwedge k.\ k \in \mathcal{T} \Longrightarrow \exists C.\ C \in \mathcal{M} \wedge k \subseteq C$

**end**

## 6.35 Absolute Retracts, Absolute Neighbourhood Retracts and Euclidean Neighbourhood Retracts

**theory** *Retracts*
**imports**
*Brouwer_Fixpoint*
*Continuous_Extension*
**begindefinition** *AR* :: *'a::topological_space set* ⇒ *bool* **where**
$AR\ S \equiv \forall U.\ \forall S'::('a * real)\ set.$
$S\ \mathit{homeomorphic}\ S' \wedge \mathit{closedin}\ (\mathit{top\_of\_set}\ U)\ S' \longrightarrow S'\ \mathit{retract\_of}\ U$

**definition** *ANR* :: *'a::topological_space set* ⇒ *bool* **where**
$ANR\ S \equiv \forall U.\ \forall S'::('a * real)\ set.$
$S\ \mathit{homeomorphic}\ S' \wedge \mathit{closedin}\ (\mathit{top\_of\_set}\ U)\ S'$
$\longrightarrow (\exists T.\ \mathit{openin}\ (\mathit{top\_of\_set}\ U)\ T \wedge S'\ \mathit{retract\_of}\ T)$

**definition** *ENR* :: *'a::topological_space set* ⇒ *bool* **where**

*ENR S* ≡ ∃ *U*. *open U* ∧ *S retract_of U*

**corollary** *ANR_imp_absolute_neighbourhood_retract*:
**fixes** *S* :: *'a::euclidean_space set* **and** *S'* :: *'b::euclidean_space set*
**assumes** *ANR S S homeomorphic S'*
**and** *clo*: *closedin* (*top_of_set U*) *S'*
**obtains** *V* **where** *openin* (*top_of_set U*) *V S' retract_of V*

**corollary** *ANR_imp_absolute_neighbourhood_retract_UNIV*:
**fixes** *S* :: *'a::euclidean_space set* **and** *S'* :: *'b::euclidean_space set*
**assumes** *ANR S* **and** *hom*: *S homeomorphic S'* **and** *clo*: *closed S'*
**obtains** *V* **where** *open V S' retract_of V*

**corollary** *neighbourhood_extension_into_ANR*:
**fixes** *f* :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
**assumes** *contf*: *continuous_on S f* **and** *fim*: *f* ∈ *S* → *T* **and** *ANR T closed S*
**obtains** *V g* **where** *S* ⊆ *V open V continuous_on V g*
*g* ∈ *V* → *T* ⋀*x*. *x* ∈ *S* ⟹ *g x* = *f x*

### 6.35.1 Analogous properties of ENRs

**corollary** *ENR_imp_absolute_neighbourhood_retract_UNIV*:
**fixes** *S* :: *'a::euclidean_space set* **and** *S'* :: *'b::euclidean_space set*
**assumes** *ENR S S homeomorphic S'*
**obtains** *T'* **where** *open T' S' retract_of T'*

**corollary** *AR_closed_Un*:
**fixes** *S* :: *'a::euclidean_space set*
**shows** ⟦*closed S*; *closed T*; *AR S*; *AR T*; *AR* (*S* ∩ *T*)⟧ ⟹ *AR* (*S* ∪ *T*)

**corollary** *ANR_closed_Un*:
**fixes** *S* :: *'a::euclidean_space set*
**shows** ⟦*closed S*; *closed T*; *ANR S*; *ANR T*; *ANR* (*S* ∩ *T*)⟧ ⟹ *ANR* (*S* ∪ *T*)

### 6.35.2 More advanced properties of ANRs and ENRs

### 6.35.3 Original ANR material, now for ENRs

### 6.35.4 Finally, spheres are ANRs and ENRs

### 6.35.5 Spheres are connected, etc

### 6.35.6 Borsuk homotopy extension theorem

**theorem** *Borsuk_homotopy_extension_homotopic*:
**fixes** $f$ :: $'a$::*euclidean_space* $\Rightarrow$ $'b$::*euclidean_space*
**assumes** *cloTS*: *closedin* (*top_of_set* $T$) $S$
**and** *anr*: ($ANR\ S \wedge ANR\ T$) $\vee$ $ANR\ U$
**and** *contf*: *continuous_on* $T\ f$
**and** $f \in T \rightarrow U$
**and** *homotopic_with_canon* ($\lambda x.\ True$) $S\ U\ f\ g$
**obtains** $g'$ **where** *homotopic_with_canon* ($\lambda x.\ True$) $T\ U\ f\ g'$
*continuous_on* $T\ g'$ *image* $g'\ T \subseteq U$
$\bigwedge x.\ x \in S \Longrightarrow g'\ x = g\ x$

### 6.35.7 More extension theorems

### 6.35.8 The complement of a set and path-connectedness

**theorem** *connected_complement_homeomorphic_convex_compact*:
**fixes** $S$ :: $'a$::*euclidean_space set* **and** $T$ :: $'b$::*euclidean_space set*
**assumes** *hom*: $S$ *homeomorphic* $T$ **and** $T$: *convex* $T$ *compact* $T$ **and** *2*: $2 \leq DIM('a)$
**shows** *connected*($-\ S$)

**corollary** *path_connected_complement_homeomorphic_convex_compact*:
**fixes** $S$ :: $'a$::*euclidean_space set* **and** $T$ :: $'b$::*euclidean_space set*
**assumes** *hom*: $S$ *homeomorphic* $T$ *convex* $T$ *compact* $T$ $2 \leq DIM('a)$
**shows** *path_connected*($-\ S$)

**end**

## 6.36 Extending Continous Maps, Invariance of Domain, etc

**theory** *Further_Topology*
**imports** *Weierstrass_Theorems Polytope Complex_Transcendental Equivalence_Lebesgue_Henstock_Integration Retracts*
**begin**

### 6.36.1 A map from a sphere to a higher dimensional sphere is nullhomotopic

**proposition** *inessential_spheremap_lowdim_gen*:
**fixes** $f$ :: *'M::euclidean_space* ⇒ *'a::euclidean_space*
**assumes** *convex S bounded S convex T bounded T*
**and** *affST*: *aff_dim S* < *aff_dim T*
**and** *contf*: *continuous_on* (*rel_frontier S*) *f*
**and** *fim*: *f* ∈ (*rel_frontier S*) → *rel_frontier T*
**obtains** *c* **where** *homotopic_with_canon* (λ*z. True*) (*rel_frontier S*) (*rel_frontier T*) *f* (λ*x. c*)

### 6.36.2 Some technical lemmas about extending maps from cell complexes

**theorem** *extend_map_cell_complex_to_sphere*:
**assumes** *finite* $\mathcal{F}$ **and** *S*: $S \subseteq \bigcup\mathcal{F}$ *closed S* **and** *T*: *convex T bounded T*
**and** *poly*: $\bigwedge X.\ X \in \mathcal{F} \Longrightarrow$ *polytope X*
**and** *aff*: $\bigwedge X.\ X \in \mathcal{F} \Longrightarrow$ *aff_dim X* < *aff_dim T*
**and** *face*: $\bigwedge X\ Y.\ [\![X \in \mathcal{F};\ Y \in \mathcal{F}]\!] \Longrightarrow (X \cap Y)$ *face_of X*
**and** *contf*: *continuous_on S f* **and** *fim*: *f* ∈ *S* → *rel_frontier T*
**obtains** *g* **where** *continuous_on* $(\bigcup\mathcal{F})$ *g*
*g* ‘ $(\bigcup\mathcal{F}) \subseteq$ *rel_frontier T* $\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$

**theorem** *extend_map_cell_complex_to_sphere_cofinite*:
**assumes** *finite* $\mathcal{F}$ **and** *S*: $S \subseteq \bigcup\mathcal{F}$ *closed S* **and** *T*: *convex T bounded T*
**and** *poly*: $\bigwedge X.\ X \in \mathcal{F} \Longrightarrow$ *polytope X*
**and** *aff*: $\bigwedge X.\ X \in \mathcal{F} \Longrightarrow$ *aff_dim X* ≤ *aff_dim T*
**and** *face*: $\bigwedge X\ Y.\ [\![X \in \mathcal{F};\ Y \in \mathcal{F}]\!] \Longrightarrow (X \cap Y)$ *face_of X*
**and** *contf*: *continuous_on S f* **and** *fim*: *f* ∈ *S* → *rel_frontier T*
**obtains** *C g* **where** *finite C disjnt C S continuous_on* $(\bigcup\mathcal{F} - C)$ *g*
*g* ‘ $(\bigcup\mathcal{F} - C) \subseteq$ *rel_frontier T* $\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$

### 6.36.3 Special cases and corollaries involving spheres

**proposition** *extend_map_affine_to_sphere_cofinite_simple*:
**fixes** $f$ :: $'a$::*euclidean_space* $\Rightarrow$ $'b$::*euclidean_space*
**assumes** *compact S convex U bounded U*
**and** *aff*: $aff\_dim\ T \leq aff\_dim\ U$
**and** $S \subseteq T$ **and** *contf*: *continuous_on S f*
**and** *fim*: $f \in S \rightarrow rel\_frontier\ U$
**obtains** *K g* **where** *finite K* $K \subseteq T$ *disjnt K S continuous_on* $(T - K)$ *g*
$g \in (T - K) \rightarrow rel\_frontier\ U$
$\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$

### 6.36.4 Extending maps to spheres

**proposition** *extend_map_affine_to_sphere_cofinite_gen*:
**fixes** $f$ :: $'a$::*euclidean_space* $\Rightarrow$ $'b$::*euclidean_space*
**assumes** *SUT*: *compact S convex U bounded U affine T* $S \subseteq T$
**and** *aff*: $aff\_dim\ T \leq aff\_dim\ U$
**and** *contf*: *continuous_on S f*
**and** *fim*: $f \in S \rightarrow rel\_frontier\ U$
**and** *dis*: $\bigwedge C.\ [\![C \in components(T - S);\ bounded\ C]\!] \Longrightarrow C \cap L \neq \{\}$
**obtains** *K g* **where** *finite K* $K \subseteq L$ $K \subseteq T$ *disjnt K S continuous_on* $(T - K)$
*g*
$g \in (T - K) \rightarrow rel\_frontier\ U$
$\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$

**corollary** *extend_map_affine_to_sphere_cofinite*:
**fixes** $f$ :: $'a$::*euclidean_space* $\Rightarrow$ $'b$::*euclidean_space*
**assumes** *SUT*: *compact S affine T* $S \subseteq T$
**and** *aff*: $aff\_dim\ T \leq DIM('b)$ **and** $0 \leq r$
**and** *contf*: *continuous_on S f*
**and** *fim*: $f \in S \rightarrow sphere\ a\ r$
**and** *dis*: $\bigwedge C.\ [\![C \in components(T - S);\ bounded\ C]\!] \Longrightarrow C \cap L \neq \{\}$
**obtains** *K g* **where** *finite K* $K \subseteq L$ $K \subseteq T$ *disjnt K S continuous_on* $(T - K)$
*g*
$g \in (T - K) \rightarrow sphere\ a\ r$ $\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$

**corollary** *extend_map_UNIV_to_sphere_cofinite*:
**fixes** $f$ :: $'a$::*euclidean_space* $\Rightarrow$ $'b$::*euclidean_space*
**assumes** $DIM('a) \leq DIM('b)$ **and** $0 \leq r$
**and** *compact S*
**and** *continuous_on S f*
**and** $f \in S \rightarrow sphere\ a\ r$

**and** $\bigwedge C.\ [\![C \in components(- S);\ bounded\ C]\!] \Longrightarrow C \cap L \neq \{\}$
**obtains** $K\ g$ **where** $finite\ K\ K \subseteq L\ disjnt\ K\ S\ continuous\_on\ (- K)\ g$
$g \in (- K) \to sphere\ a\ r\ \bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$

**corollary** *extend_map_UNIV_to_sphere_no_bounded_component*:
**fixes** $f :: 'a::euclidean\_space \Rightarrow 'b::euclidean\_space$
**assumes** $aff$: $DIM('a) \leq DIM('b)$ **and** $0 \leq r$
**and** $SUT$: $compact\ S$
**and** $contf$: $continuous\_on\ S\ f$
**and** $fim$: $f \in S \to sphere\ a\ r$
**and** $dis$: $\bigwedge C.\ C \in components(- S) \Longrightarrow \neg\ bounded\ C$
**obtains** $g$ **where** $continuous\_on\ UNIV\ g\ g \in UNIV \to sphere\ a\ r\ \bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$

**theorem** *Borsuk_separation_theorem_gen*:
**fixes** $S :: 'a::euclidean\_space\ set$
**assumes** $compact\ S$
**shows** $(\forall c \in components(- S).\ \neg bounded\ c) \longleftrightarrow$
$(\forall f.\ continuous\_on\ S\ f \land f \in S \to sphere\ (0::'a)\ 1$
$\longrightarrow (\exists c.\ homotopic\_with\_canon\ (\lambda x.\ True)\ S\ (sphere\ 0\ 1)\ f\ (\lambda x.\ c)))$
(**is** $?lhs = ?rhs$)

**corollary** *Borsuk_separation_theorem*:
**fixes** $S :: 'a::euclidean\_space\ set$
**assumes** $compact\ S$ **and** $2$: $2 \leq DIM('a)$
**shows** $connected(- S) \longleftrightarrow$
$(\forall f.\ continuous\_on\ S\ f \land f \in S \to sphere\ (0::'a)\ 1$
$\longrightarrow (\exists c.\ homotopic\_with\_canon\ (\lambda x.\ True)\ S\ (sphere\ 0\ 1)\ f\ (\lambda x.\ c)))$
(**is** $?lhs = ?rhs$)

**proposition** *Jordan_Brouwer_separation*:
**fixes** $S :: 'a::euclidean\_space\ set$ **and** $a::'a$
**assumes** $hom$: $S\ homeomorphic\ sphere\ a\ r$ **and** $0 < r$
**shows** $\neg\ connected(- S)$

**proposition** *Jordan_Brouwer_frontier*:
**fixes** $S :: 'a::euclidean\_space\ set$ **and** $a::'a$
**assumes** $S$: $S\ homeomorphic\ sphere\ a\ r$ **and** $T$: $T \in components(- S)$ **and** $2$: $2 \leq DIM('a)$
**shows** $frontier\ T = S$

**proposition** *Jordan_Brouwer_nonseparation*:
**fixes** $S :: 'a::euclidean\_space\ set$ **and** $a::'a$

**assumes** *S*: *S homeomorphic sphere a r* **and** $T \subset S$ **and** *2*: $2 \leq DIM('a)$
**shows** *connected*$(- T)$

### 6.36.5 Invariance of domain and corollaries

**theorem** *invariance_of_domain*:
**fixes** $f$ :: $'a \Rightarrow 'a$::*euclidean_space*
**assumes** *continuous_on S f open S inj_on f S*
**shows** *open*(*f* ‘ *S*)

**corollary** *invariance_of_domain_subspaces*:
**fixes** $f$ :: $'a$::*euclidean_space* ⇒ $'b$::*euclidean_space*
**assumes** *ope*: *openin* (*top_of_set U*) *S*
**and** *subspace U subspace V* **and** *VU*: $dim\ V \leq dim\ U$
**and** *contf*: *continuous_on S f* **and** *fim*: $f$ ‘ $S \subseteq V$
**and** *injf*: *inj_on f S*
**shows** *openin* (*top_of_set V*) (*f* ‘ *S*)

**corollary** *invariance_of_dimension_subspaces*:
**fixes** $f$ :: $'a$::*euclidean_space* ⇒ $'b$::*euclidean_space*
**assumes** *ope*: *openin* (*top_of_set U*) *S*
**and** *subspace U subspace V*
**and** *contf*: *continuous_on S f* **and** *fim*: $f$ ‘ $S \subseteq V$
**and** *injf*: *inj_on f S* **and** $S \neq \{\}$
**shows** $dim\ U \leq dim\ V$

**corollary** *invariance_of_domain_affine_sets*:
**fixes** $f$ :: $'a$::*euclidean_space* ⇒ $'b$::*euclidean_space*
**assumes** *ope*: *openin* (*top_of_set U*) *S*
**and** *aff*: *affine U affine V* $aff\_dim\ V \leq aff\_dim\ U$
**and** *contf*: *continuous_on S f* **and** *fim*: $f$ ‘ $S \subseteq V$
**and** *injf*: *inj_on f S*
**shows** *openin* (*top_of_set V*) (*f* ‘ *S*)

**corollary** *invariance_of_dimension_affine_sets*:
**fixes** $f$ :: $'a$::*euclidean_space* ⇒ $'b$::*euclidean_space*
**assumes** *ope*: *openin* (*top_of_set U*) *S*
**and** *aff*: *affine U affine V*
**and** *contf*: *continuous_on S f* **and** *fim*: $f$ ‘ $S \subseteq V$
**and** *injf*: *inj_on f S* **and** $S \neq \{\}$
**shows** $aff\_dim\ U \leq aff\_dim\ V$

**corollary** *invariance_of_dimension*:
**fixes** $f$ :: $'a$::*euclidean_space* ⇒ $'b$::*euclidean_space*
**assumes** *contf*: *continuous_on S f* **and** *open S*
**and** *injf*: *inj_on f S* **and** $S \neq \{\}$
**shows** $DIM('a) \leq DIM('b)$

**corollary** *continuous_injective_image_subspace_dim_le*:
**fixes** $f$ :: $'a$::*euclidean_space* ⇒ $'b$::*euclidean_space*
**assumes** *subspace S subspace T*
**and** *contf*: *continuous_on S f* **and** *fim*: $f$ ‘ $S \subseteq T$
**and** *injf*: *inj_on f S*
**shows** *dim* $S \leq$ *dim* $T$

**corollary** *invariance_of_domain_homeomorphic*:
**fixes** $f$ :: $'a$::*euclidean_space* ⇒ $'b$::*euclidean_space*
**assumes** *open S continuous_on S f* $DIM('b) \leq DIM('a)$ *inj_on f S*
**shows** *S homeomorphic* ($f$ ‘ $S$)

**proposition** *homeomorphic_interiors*:
**fixes** $S$ :: $'a$::*euclidean_space set* **and** $T$ :: $'b$::*euclidean_space set*
**assumes** *S homeomorphic T interior* $S = \{\} \longleftrightarrow$ *interior* $T = \{\}$
**shows** (*interior S*) *homeomorphic* (*interior T*)

**proposition** *uniformly_continuous_homeomorphism_UNIV_trivial*:
**fixes** $f$ :: $'a$::*euclidean_space* ⇒ $'a$
**assumes** *contf*: *uniformly_continuous_on S f* **and** *hom*: *homeomorphism S UNIV f g*
**shows** $S$ = *UNIV*

### 6.36.6 Formulation of loop homotopy in terms of maps out of type complex

**proposition** *simply_connected_eq_homotopic_circlemaps*:
**fixes** $S$ :: $'a$::*real_normed_vector set*
**shows** *simply_connected* $S \longleftrightarrow$
(∀ *f g*::*complex* ⇒ $'a$.
*continuous_on* (*sphere 0 1*) $f \wedge f$ ‘ (*sphere 0 1*) $\subseteq S \wedge$
*continuous_on* (*sphere 0 1*) $g \wedge g$ ‘ (*sphere 0 1*) $\subseteq S$
⟶ *homotopic_with_canon* (λ*h*. *True*) (*sphere 0 1*) *S f g*)

**proposition** *simply_connected_eq_contractible_circlemap*:
**fixes** $S$ :: $'a$::*real_normed_vector set*
**shows** *simply_connected* $S \longleftrightarrow$
*path_connected* $S \wedge$
(∀ *f*::*complex* ⇒ $'a$.
*continuous_on* (*sphere 0 1*) $f \wedge f$ ‘(*sphere 0 1*) $\subseteq S$

$\longrightarrow (\exists\, a.\ homotopic\_with\_canon\ (\lambda h.\ True)\ (sphere\ 0\ 1)\ S\ f\ (\lambda x.\ a)))$

**corollary** *homotopy_eqv_simple_connectedness*:
**fixes** $S$ :: *'a::real_normed_vector set* **and** $T$ :: *'b::real_normed_vector set*
**shows** $S$ *homotopy_eqv* $T \Longrightarrow$ *simply_connected* $S \longleftrightarrow$ *simply_connected* $T$

### 6.36.7 Homeomorphism of simple closed curves to circles

**proposition** *homeomorphic_simple_path_image_circle*:
**fixes** $a$ :: *complex* **and** $\gamma$ :: *real* $\Rightarrow$ *'a::t2_space*
**assumes** *simple_path* $\gamma$ **and** *loop*: *pathfinish* $\gamma$ = *pathstart* $\gamma$ **and** $0 < r$
**shows** (*path_image* $\gamma$) *homeomorphic sphere* $a$ $r$

### 6.36.8 Dimension-based conditions for various homeomorphisms

### 6.36.9 more invariance of domain

**proposition** *invariance_of_domain_sphere_affine_set_gen*:
**fixes** $f$ :: *'a::euclidean_space* $\Rightarrow$ *'b::euclidean_space*
**assumes** *contf*: *continuous_on* $S$ $f$ **and** *injf*: *inj_on* $f$ $S$ **and** *fim*: $f$ ‘ $S \subseteq T$
**and** $U$: *bounded* $U$ *convex* $U$
**and** *affine* $T$ **and** *affTU*: *aff_dim* $T$ < *aff_dim* $U$
**and** *ope*: *openin* (*top_of_set* (*rel_frontier* $U$)) $S$
**shows** *openin* (*top_of_set* $T$) ($f$ ‘ $S$)

**proposition** *simply_connected_punctured_convex*:
**fixes** $a$ :: *'a::euclidean_space*
**assumes** *convex* $S$ **and** *3*: $3 \le$ *aff_dim* $S$
**shows** *simply_connected*($S - \{a\}$)

**corollary** *simply_connected_punctured_universe*:
**fixes** $a$ :: *'a::euclidean_space*
**assumes** $3 \le DIM('a)$
**shows** *simply_connected*($- \{a\}$)

### 6.36.10 The power, squaring and exponential functions as covering maps

**proposition** *covering_space_power_punctured_plane*:
**assumes** $0 < n$
**shows** *covering_space* ($- \{0\}$) ($\lambda z$::*complex*. $z$^$n$) ($- \{0\}$)

**corollary** *covering_space_square_punctured_plane*:

*covering_space* $(- \{0\})$ $(\lambda z::complex.\ z^2)$ $(- \{0\})$

**proposition** *covering_space_exp_punctured_plane*:
*covering_space UNIV* $(\lambda z::complex.\ exp\ z)$ $(- \{0\})$

### 6.36.11 Hence the Borsukian results about mappings into circles

**corollary** *inessential_imp_continuous_logarithm_circle*:
**fixes** $f :: 'a::real\_normed\_vector \Rightarrow complex$
**assumes** *homotopic_with_canon* $(\lambda h.\ True)$ $S$ $(sphere\ 0\ 1)$ $f$ $(\lambda t.\ a)$
**obtains** $g$ **where** *continuous_on* $S\ g$ **and** $\bigwedge x.\ x \in S \Longrightarrow f\ x = exp(g\ x)$

**proposition** *homotopic_with_sphere_times*:
**fixes** $f :: 'a::real\_normed\_vector \Rightarrow complex$
**assumes** *hom*: *homotopic_with_canon* $(\lambda x.\ True)$ $S$ $(sphere\ 0\ 1)$ $f\ g$ **and** *conth*: *continuous_on* $S\ h$
**and** *hin*: $\bigwedge x.\ x \in S \Longrightarrow h\ x \in sphere\ 0\ 1$
**shows** *homotopic_with_canon* $(\lambda x.\ True)$ $S$ $(sphere\ 0\ 1)$ $(\lambda x.\ f\ x * h\ x)$ $(\lambda x.\ g\ x * h\ x)$

**proposition** *homotopic_circlemaps_divide*:
**fixes** $f :: 'a::real\_normed\_vector \Rightarrow complex$
**shows** *homotopic_with_canon* $(\lambda x.\ True)$ $S$ $(sphere\ 0\ 1)$ $f\ g \longleftrightarrow$
*continuous_on* $S\ f \wedge f\ `\ S \subseteq sphere\ 0\ 1\ \wedge$
*continuous_on* $S\ g \wedge g\ `\ S \subseteq sphere\ 0\ 1\ \wedge$
$(\exists c.$ *homotopic_with_canon* $(\lambda x.\ True)$ $S$ $(sphere\ 0\ 1)$ $(\lambda x.\ f\ x\ /\ g\ x)$ $(\lambda x.\ c))$

### 6.36.12 Upper and lower hemicontinuous functions

**proposition** *upper_lower_hemicontinuous_explicit*:
**fixes** $T :: ('b::\{real\_normed\_vector,heine\_borel\})\ set$
**assumes** *fST*: $\bigwedge x.\ x \in S \Longrightarrow f\ x \subseteq T$
**and** *ope*: $\bigwedge U.$ *openin* $(top\_of\_set\ T)\ U$
$\Longrightarrow$ *openin* $(top\_of\_set\ S)$ $\{x \in S.\ f\ x \subseteq U\}$
**and** *clo*: $\bigwedge U.$ *closedin* $(top\_of\_set\ T)\ U$
$\Longrightarrow$ *closedin* $(top\_of\_set\ S)$ $\{x \in S.\ f\ x \subseteq U\}$
**and** $x \in S$ $0 < e$ **and** *bofx*: $bounded(f\ x)$ **and** *fx_ne*: $f\ x \neq \{\}$
**obtains** $d$ **where** $0 < d$
$\bigwedge x'.\ [\![x' \in S;\ dist\ x\ x' < d]\!]$
$\Longrightarrow (\forall y \in f\ x.\ \exists y'.\ y' \in f\ x' \wedge dist\ y\ y' < e) \wedge$
$(\forall y' \in f\ x'.\ \exists y.\ y \in f\ x \wedge dist\ y'\ y < e)$

### 6.36.13 Complex logs exist on various "well-behaved" sets

### 6.36.14 Another simple case where sphere maps are nullhomotopic

### 6.36.15 Holomorphic logarithms and square roots

### 6.36.16 The "Borsukian" property of sets

**definition** *Borsukian* **where**
*Borsukian S* ≡
∀*f*. *continuous_on S f* ∧ *f* ∈ *S* → (− {*0*::*complex*})
⟶ (∃ *a*. *homotopic_with_canon* (λ*h*. *True*) *S* (− {*0*}) *f* (λ*x*. *a*))

**proposition** *Borsukian_sphere*:
**fixes** *a* :: ′*a*::*euclidean_space*
**shows** *3* ≤ *DIM*(′*a*) ⟹ *Borsukian* (*sphere a r*)

**proposition** *Borsukian_open_Un*:
**fixes** *S* :: ′*a*::*real_normed_vector set*
**assumes** *opeS*: *openin* (*top_of_set* (*S* ∪ *T*)) *S*
**and** *opeT*: *openin* (*top_of_set* (*S* ∪ *T*)) *T*
**and** *BS*: *Borsukian S* **and** *BT*: *Borsukian T* **and** *ST*: *connected*(*S* ∩ *T*)
**shows** *Borsukian*(*S* ∪ *T*)

**proposition** *closed_irreducible_separator*:
**fixes** *a* :: ′*a*::*real_normed_vector*
**assumes** *closed S* **and** *ab*: ¬ *connected_component* (− *S*) *a b*
**obtains** *T* **where** *T* ⊆ *S closed T T* ≠ {} ¬ *connected_component* (− *T*) *a b*
⋀*U*. *U* ⊂ *T* ⟹ *connected_component* (− *U*) *a b*

### 6.36.17 Unicoherence (closed)

**definition** *unicoherent* **where**
*unicoherent U* ≡
∀ *S T*. *connected S* ∧ *connected T* ∧ *S* ∪ *T* = *U* ∧
*closedin* (*top_of_set U*) *S* ∧ *closedin* (*top_of_set U*) *T*
⟶ *connected* (*S* ∩ *T*)

**proposition** *homeomorphic_unicoherent*:
**assumes** *ST*: *S homeomorphic T* **and** *S*: *unicoherent S*
**shows** *unicoherent T*

**corollary** *contractible_imp_unicoherent*:
**fixes** *U* :: *'a::euclidean_space set*
**assumes** *contractible U* **shows** *unicoherent U*

**corollary** *convex_imp_unicoherent*:
**fixes** *U* :: *'a::euclidean_space set*
**assumes** *convex U* **shows** *unicoherent U*
**corollary** *unicoherent_UNIV*: *unicoherent* (*UNIV* :: *'a* :: *euclidean_space set*)

### 6.36.18 Several common variants of unicoherence

### 6.36.19 Some separation results

**proposition** *separation_by_component_open*:
**fixes** *S* :: *'a* :: *euclidean_space set*
**assumes** *open S* **and** *non*: ¬ *connected*(− *S*)
**obtains** *C* **where** *C* ∈ *components S* ¬ *connected*(− *C*)

**proposition** *inessential_eq_extensible*:
**fixes** *f* :: *'a::euclidean_space* ⇒ *complex*
**assumes** *closed S*
**shows** (∃ *a*. *homotopic_with_canon* (λ*h*. *True*) *S* (−{*0*}) *f* (λ*t*. *a*)) ⟷
(∃ *g*. *continuous_on UNIV g* ∧ (∀ *x* ∈ *S*. *g x* = *f x*) ∧ (∀ *x*. *g x* ≠ *0*))
(**is** *?lhs* = *?rhs*)

**proposition** *Janiszewski_dual*:
**fixes** *S* :: *complex set*
**assumes**
*compact S compact T connected S connected T connected*(− (*S* ∪ *T*))
**shows** *connected*(*S* ∩ *T*)

**end**

## 6.37 The Jordan Curve Theorem and Applications

**theory** *Jordan_Curve*
**imports** *Arcwise_Connected Further_Topology*
**begin**

### 6.37.1 Janiszewski's theorem

**theorem** *Janiszewski*:
**fixes** *a b* :: *complex*
**assumes** *compact S closed T* **and** *conST*: *connected* (*S* ∩ *T*)
**and** *ccS*: *connected_component* (− *S*) *a b* **and** *ccT*: *connected_component* (− *T*) *a b*
**shows** *connected_component* (− (*S* ∪ *T*)) *a b*

### 6.37.2 The Jordan Curve theorem

**corollary** *Jordan_inside_outside*:
**fixes** *c* :: *real* ⇒ *complex*
**assumes** *simple_path c pathfinish c* = *pathstart c*
**shows** *inside*(*path_image c*) ≠ {} ∧
*open*(*inside*(*path_image c*)) ∧
*connected*(*inside*(*path_image c*)) ∧
*outside*(*path_image c*) ≠ {} ∧
*open*(*outside*(*path_image c*)) ∧
*connected*(*outside*(*path_image c*)) ∧
*bounded*(*inside*(*path_image c*)) ∧
¬ *bounded*(*outside*(*path_image c*)) ∧
*inside*(*path_image c*) ∩ *outside*(*path_image c*) = {} ∧
*inside*(*path_image c*) ∪ *outside*(*path_image c*) =
− *path_image c* ∧
*frontier*(*inside*(*path_image c*)) = *path_image c* ∧
*frontier*(*outside*(*path_image c*)) = *path_image c*

**theorem** *split_inside_simple_closed_curve*:
**fixes** *c* :: *real* ⇒ *complex*
**assumes** *simple_path c1* **and** *c1*: *pathstart c1* = *a pathfinish c1* = *b*
**and** *simple_path c2* **and** *c2*: *pathstart c2* = *a pathfinish c2* = *b*
**and** *simple_path c* **and** *c*: *pathstart c* = *a pathfinish c* = *b*
**and** *a* ≠ *b*
**and** *c1c2*: *path_image c1* ∩ *path_image c2* = {*a*,*b*}
**and** *c1c*: *path_image c1* ∩ *path_image c* = {*a*,*b*}
**and** *c2c*: *path_image c2* ∩ *path_image c* = {*a*,*b*}
**and** *ne_12*: *path_image c* ∩ *inside*(*path_image c1* ∪ *path_image c2*) ≠ {}
**obtains** *inside*(*path_image c1* ∪ *path_image c*) ∩ *inside*(*path_image c2* ∪ *path_image c*) = {}
*inside*(*path_image c1* ∪ *path_image c*) ∪ *inside*(*path_image c2* ∪ *path_image c*) ∪
(*path_image c* − {*a*,*b*}) = *inside*(*path_image c1* ∪ *path_image c2*)

**end**

## 6.38 Polynomial Functions: Extremal Behaviour and Root Counts

**theory** *Poly_Roots*
**imports** *Complex_Main*
**begin**

### 6.38.1 Basics about polynomial functions: extremal behaviour and root counts

**proposition** *polyfun_extremal_lemma*:
**fixes** $c :: nat \Rightarrow 'a::real\_normed\_div\_algebra$
**assumes** $e > 0$
**shows** $\exists M. \forall z. M \leq norm\ z \longrightarrow norm(\sum i \leq n.\ c\ i * z\hat{\ }i) \leq e * norm(z)\ \hat{\ }\ Suc\ n$

**proposition** *polyfun_extremal*:
**fixes** $c :: nat \Rightarrow 'a::real\_normed\_div\_algebra$
**assumes** $\exists k.\ k \neq 0 \wedge k \leq n \wedge c\ k \neq 0$
**shows** $eventually\ (\lambda z.\ norm(\sum i \leq n.\ c\ i * z\hat{\ }i) \geq B)\ at\_infinity$

**proposition** *polyfun_rootbound*:
**fixes** $c :: nat \Rightarrow 'a::\{comm\_ring,real\_normed\_div\_algebra\}$
**assumes** $\exists k.\ k \leq n \wedge c\ k \neq 0$
**shows** $finite\ \{z.\ (\sum i \leq n.\ c\ i * z\hat{\ }i) = 0\} \wedge card\ \{z.\ (\sum i \leq n.\ c\ i * z\hat{\ }i) = 0\} \leq n$

**corollary**
**fixes** $c :: nat \Rightarrow 'a::\{comm\_ring,real\_normed\_div\_algebra\}$
**assumes** $\exists k.\ k \leq n \wedge c\ k \neq 0$
**shows** *polyfun_rootbound_finite*: $finite\ \{z.\ (\sum i \leq n.\ c\ i * z\hat{\ }i) = 0\}$
**and** *polyfun_rootbound_card*: $card\ \{z.\ (\sum i \leq n.\ c\ i * z\hat{\ }i) = 0\} \leq n$

**proposition** *polyfun_finite_roots*:
**fixes** $c :: nat \Rightarrow 'a::\{comm\_ring,real\_normed\_div\_algebra\}$
**shows** $finite\ \{z.\ (\sum i \leq n.\ c\ i * z\hat{\ }i) = 0\} \longleftrightarrow (\exists k.\ k \leq n \wedge c\ k \neq 0)$

**theorem** *polyfun_eq_const*:
**fixes** $c :: nat \Rightarrow 'a::\{comm\_ring,real\_normed\_div\_algebra\}$
**shows** $(\forall z.\ (\sum i \leq n.\ c\ i * z\hat{\ }i) = k) \longleftrightarrow c\ 0 = k \wedge (\forall k.\ k \neq 0 \wedge k \leq n \longrightarrow c\ k = 0)$

**end**

## 6.39 Generalised Binomial Theorem

**theory** *Generalised_Binomial_Theorem*
**imports**
  *Complex_Main*
  *Complex_Transcendental*
  *Summation_Tests*
**begin**

**theorem** *gen_binomial_complex*:
  **fixes** $z$ :: *complex*
  **assumes** *norm* $z < 1$
  **shows** $(\lambda n.\ (a\ gchoose\ n) * z\hat{\ }n)$ *sums* $(1 + z)$ *powr* $a$

**end**

## 6.40 Vitali Covering Theorem and an Application to Negligibility

**theory** *Vitali_Covering_Theorem*
**imports**
  *HOL−Combinatorics.Permutations*
  *Equivalence_Lebesgue_Henstock_Integration*
**begin**

### 6.40.1 Vitali covering theorem

**theorem** *Vitali_covering_theorem_cballs*:
  **fixes** $a$ :: $'a \Rightarrow 'n$::*euclidean_space*
  **assumes** $r$: $\bigwedge i.\ i \in K \Longrightarrow 0 < r\ i$
    **and** $S$: $\bigwedge x\ d.\ [\![x \in S;\ 0 < d]\!]$
      $\Longrightarrow \exists i.\ i \in K \wedge x \in cball\ (a\ i)\ (r\ i) \wedge r\ i < d$
  **obtains** $C$ **where** *countable* $C$ $C \subseteq K$
    *pairwise* $(\lambda i\ j.\ disjnt\ (cball\ (a\ i)\ (r\ i))\ (cball\ (a\ j)\ (r\ j)))$ $C$
    *negligible*$(S - (\bigcup i \in C.\ cball\ (a\ i)\ (r\ i)))$

**theorem** *Vitali_covering_theorem_balls*:
  **fixes** $a$ :: $'a \Rightarrow 'b$::*euclidean_space*
  **assumes** $S$: $\bigwedge x\ d.\ [\![x \in S;\ 0 < d]\!] \Longrightarrow \exists i.\ i \in K \wedge x \in ball\ (a\ i)\ (r\ i) \wedge r\ i < d$
  **obtains** $C$ **where** *countable* $C$ $C \subseteq K$
    *pairwise* $(\lambda i\ j.\ disjnt\ (ball\ (a\ i)\ (r\ i))\ (ball\ (a\ j)\ (r\ j)))$ $C$
    *negligible*$(S - (\bigcup i \in C.\ ball\ (a\ i)\ (r\ i)))$

**proposition** *negligible_eq_zero_density*:
*negligible S* ⟷
(∀ *x*∈*S*. ∀ *r*>*0*. ∀ *e*>*0*. ∃ *d*. *0* < *d* ∧ *d* ≤ *r* ∧
(∃ *U*. *S* ∩ *ball x d* ⊆ *U* ∧ *U* ∈ *lmeasurable* ∧ *measure lebesgue U* < *e* ∗ *measure lebesgue* (*ball x d*)))

**end**

## 6.41 Change of Variables Theorems

**theory** *Change_Of_Vars*
**imports** *Vitali_Covering_Theorem Determinants*

**begin**

### 6.41.1 Measurable Shear and Stretch

**proposition**
**fixes** *a* :: *real*^'*n*
**assumes** *m* ≠ *n* **and** *ab_ne*: *cbox a b* ≠ {} **and** *an*: *0* ≤ *a*$*n*
**shows** *measurable_shear_interval*: (λ*x*. χ *i*. *if i* = *m then x*$*m* + *x*$*n else x*$*i*) ‘ (*cbox a b*) ∈ *lmeasurable*
(**is** *?f* ‘ _ ∈ _)
**and** *measure_shear_interval*: *measure lebesgue* ((λ*x*. χ *i*. *if i* = *m then x*$*m* + *x*$*n else x*$*i*) ‘ *cbox a b*)
= *measure lebesgue* (*cbox a b*) (**is** *?Q*)

**proposition**
**fixes** *S* :: (*real*^'*n*) *set*
**assumes** *S* ∈ *lmeasurable*
**shows** *measurable_stretch*: ((λ*x*. χ *k*. *m k* ∗ *x*$*k*) ‘ *S*) ∈ *lmeasurable* (**is** *?f* ‘ *S* ∈ _)
**and** *measure_stretch*: *measure lebesgue* ((λ*x*. χ *k*. *m k* ∗ *x*$*k*) ‘ *S*) = |*prod m UNIV*| ∗ *measure lebesgue S*
(**is** *?MEQ*)

**proposition**
**fixes** *f* :: *real*^'*n*::{*finite*,*wellorder*} ⇒ *real*^'*n*::_
**assumes** *linear f S* ∈ *lmeasurable*
**shows** *measurable_linear_image*: (*f* ‘ *S*) ∈ *lmeasurable*
**and** *measure_linear_image*: *measure lebesgue* (*f* ‘ *S*) = |*det* (*matrix f*)| ∗ *measure lebesgue S* (**is** *?Q f S*)

**proposition** *measure_semicontinuous_with_hausdist_explicit*:
**assumes** *bounded S* **and** *neg*: *negligible*(*frontier S*) **and** *e* > *0*
**obtains** *d* **where** *d* > *0*

$\bigwedge T.\ [\![T \in lmeasurable;\ \bigwedge y.\ y \in T \Longrightarrow \exists x.\ x \in S \wedge dist\ x\ y < d]\!]$
$\Longrightarrow measure\ lebesgue\ T < measure\ lebesgue\ S + e$

**proposition**
**fixes** $f$ :: *real^'n::{finite,wellorder}* ⇒ *real^'n::_*
**assumes** $S$: $S \in lmeasurable$
**and** *deriv*: $\bigwedge x.\ x \in S \Longrightarrow (f\ has\_derivative\ f'\ x)\ (at\ x\ within\ S)$
**and** *int*: $(\lambda x.\ |det\ (matrix\ (f'\ x))|)\ integrable\_on\ S$
**and** *bounded*: $\bigwedge x.\ x \in S \Longrightarrow |det\ (matrix\ (f'\ x))| \le B$
**shows** *measurable_bounded_differentiable_image*:
$f$ ‘ $S \in lmeasurable$
**and** *measure_bounded_differentiable_image*:
*measure lebesgue* ($f$ ‘ $S$) $\le B *$ *measure lebesgue* $S$ (**is** *?M*)

**theorem**
**fixes** $f$ :: *real^'n::{finite,wellorder}* ⇒ *real^'n::_*
**assumes** $S$: $S \in$ *sets lebesgue*
**and** *deriv*: $\bigwedge x.\ x \in S \Longrightarrow (f\ has\_derivative\ f'\ x)\ (at\ x\ within\ S)$
**and** *int*: $(\lambda x.\ |det\ (matrix\ (f'\ x))|)\ integrable\_on\ S$
**shows** *measurable_differentiable_image*: $f$ ‘ $S \in lmeasurable$
**and** *measure_differentiable_image*:
*measure lebesgue* ($f$ ‘ $S$) $\le$ *integral* $S$ $(\lambda x.\ |det\ (matrix\ (f'\ x))|)$ (**is** *?M*)

### 6.41.2 Borel measurable Jacobian determinant

**proposition** *borel_measurable_partial_derivatives*:
**fixes** $f$ :: *real^'m::{finite,wellorder}* ⇒ *real^'n*
**assumes** $S$: $S \in$ *sets lebesgue*
**and** $f$: $\bigwedge x.\ x \in S \Longrightarrow (f\ has\_derivative\ f'\ x)\ (at\ x\ within\ S)$
**shows** $(\lambda x.\ (matrix(f'\ x)\$m\$n)) \in$ *borel_measurable* (*lebesgue_on* $S$)

**theorem** *borel_measurable_det_Jacobian*:
**fixes** $f$ :: *real^'n::{finite,wellorder}* ⇒ *real^'n::_*
**assumes** $S$: $S \in$ *sets lebesgue* **and** $f$: $\bigwedge x.\ x \in S \Longrightarrow (f\ has\_derivative\ f'\ x)\ (at\ x\ within\ S)$
**shows** $(\lambda x.\ det(matrix(f'\ x))) \in$ *borel_measurable* (*lebesgue_on* $S$)
**theorem** *borel_measurable_lebesgue_on_preimage_borel*:
**fixes** $f$ :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
**assumes** $S \in$ *sets lebesgue*
**shows** $f \in$ *borel_measurable* (*lebesgue_on* $S$) $\longleftrightarrow$
$(\forall T.\ T \in sets\ borel \longrightarrow \{x \in S.\ f\ x \in T\} \in sets\ lebesgue)$

### 6.41.3 Simplest case of Sard's theorem (we don't need continuity of derivative)

**theorem** *baby_Sard*:
**fixes** $f$ :: *real^'m::{finite,wellorder}* ⇒ *real^'n::{finite,wellorder}*
**assumes** *mlen*: $CARD('m) \leq CARD('n)$
**and** *der*: $\bigwedge x.\ x \in S \Longrightarrow (f\ has\_derivative\ f'\ x)\ (at\ x\ within\ S)$
**and** *rank*: $\bigwedge x.\ x \in S \Longrightarrow rank(matrix(f'\ x)) < CARD('n)$
**shows** *negligible*($f$ ‘ $S$)

### 6.41.4 A one-way version of change-of-variables not assuming injectivity.

**proposition** *absolutely_integrable_on_image*:
**fixes** $f$ :: *real^'m::{finite,wellorder}* ⇒ *real^'n* **and** $g$ :: *real^'m::_* ⇒ *real^'m::_*
**assumes** *der_g*: $\bigwedge x.\ x \in S \Longrightarrow (g\ has\_derivative\ g'\ x)\ (at\ x\ within\ S)$
**and** *intS*: $(\lambda x.\ |det\ (matrix\ (g'\ x))| *_R f(g\ x))$ *absolutely_integrable_on* $S$
**shows** $f$ *absolutely_integrable_on* ($g$ ‘ $S$)

**proposition** *integral_on_image_ubound*:
**fixes** $f$ :: *real^'n::{finite,wellorder}* ⇒ *real* **and** $g$ :: *real^'n::_* ⇒ *real^'n::_*
**assumes**$\bigwedge x.\ x \in S \Longrightarrow 0 \leq f(g\ x)$
**and** $\bigwedge x.\ x \in S \Longrightarrow (g\ has\_derivative\ g'\ x)\ (at\ x\ within\ S)$
**and** $(\lambda x.\ |det\ (matrix\ (g'\ x))| * f(g\ x))$ *integrable_on* $S$
**shows** *integral* ($g$ ‘ $S$) $f \leq$ *integral* $S$ $(\lambda x.\ |det\ (matrix\ (g'\ x))| * f(g\ x))$

### 6.41.5 Change-of-variables theorem

**theorem** *has_absolute_integral_change_of_variables_invertible*:
**fixes** $f$ :: *real^'m::{finite,wellorder}* ⇒ *real^'n* **and** $g$ :: *real^'m::_* ⇒ *real^'m::_*
**assumes** *der_g*: $\bigwedge x.\ x \in S \Longrightarrow (g\ has\_derivative\ g'\ x)\ (at\ x\ within\ S)$
**and** *hg*: $\bigwedge x.\ x \in S \Longrightarrow h(g\ x) = x$
**and** *conth*: *continuous_on* ($g$ ‘ $S$) $h$
**shows** $(\lambda x.\ |det\ (matrix\ (g'\ x))| *_R f(g\ x))$ *absolutely_integrable_on* $S$ ∧ *integral* $S$ $(\lambda x.\ |det\ (matrix\ (g'\ x))| *_R f(g\ x)) = b \longleftrightarrow$
$f$ *absolutely_integrable_on* ($g$ ‘ $S$) ∧ *integral* ($g$ ‘ $S$) $f = b$

(**is** *?lhs* = *?rhs*)

**theorem** *has_absolute_integral_change_of_variables_compact*:
**fixes** $f$ :: *real*^'*m*::{*finite*,*wellorder*} ⇒ *real*^'*n* **and** $g$ :: *real*^'*m*::_ ⇒ *real*^'*m*::_
**assumes** *compact S*
**and** *der_g*: $\bigwedge x.\ x \in S \Longrightarrow$ (*g has_derivative g' x*) (*at x within S*)
**and** *inj*: *inj_on g S*
**shows** ((λ*x*. |*det* (*matrix* (*g' x*))| $*_R$ *f*(*g x*)) *absolutely_integrable_on S* ∧
*integral S* (λ*x*. |*det* (*matrix* (*g' x*))| $*_R$ *f*(*g x*)) = *b*)
⟷ *f absolutely_integrable_on* (*g* ‘ *S*) ∧ *integral* (*g* ‘ *S*) *f* = *b*)

**theorem** *has_absolute_integral_change_of_variables*:
**fixes** $f$ :: *real*^'*m*::{*finite*,*wellorder*} ⇒ *real*^'*n* **and** $g$ :: *real*^'*m*::_ ⇒ *real*^'*m*::_
**assumes** *S*: *S* ∈ *sets lebesgue*
**and** *der_g*: $\bigwedge x.\ x \in S \Longrightarrow$ (*g has_derivative g' x*) (*at x within S*)
**and** *inj*: *inj_on g S*
**shows** (λ*x*. |*det* (*matrix* (*g' x*))| $*_R$ *f*(*g x*)) *absolutely_integrable_on S* ∧
*integral S* (λ*x*. |*det* (*matrix* (*g' x*))| $*_R$ *f*(*g x*)) = *b*
⟷ *f absolutely_integrable_on* (*g* ‘ *S*) ∧ *integral* (*g* ‘ *S*) *f* = *b*

**corollary** *absolutely_integrable_change_of_variables*:
**fixes** $f$ :: *real*^'*m*::{*finite*,*wellorder*} ⇒ *real*^'*n* **and** $g$ :: *real*^'*m*::_ ⇒ *real*^'*m*::_
**assumes** *S* ∈ *sets lebesgue*
**and** $\bigwedge x.\ x \in S \Longrightarrow$ (*g has_derivative g' x*) (*at x within S*)
**and** *inj_on g S*
**shows** *f absolutely_integrable_on* (*g* ‘ *S*)
⟷ (λ*x*. |*det* (*matrix* (*g' x*))| $*_R$ *f*(*g x*)) *absolutely_integrable_on S*

**corollary** *integral_change_of_variables*:
**fixes** $f$ :: *real*^'*m*::{*finite*,*wellorder*} ⇒ *real*^'*n* **and** $g$ :: *real*^'*m*::_ ⇒ *real*^'*m*::_
**assumes** *S*: *S* ∈ *sets lebesgue*
**and** *der_g*: $\bigwedge x.\ x \in S \Longrightarrow$ (*g has_derivative g' x*) (*at x within S*)
**and** *inj*: *inj_on g S*
**and** *disj*: (*f absolutely_integrable_on* (*g* ‘ *S*) ∨
(λ*x*. |*det* (*matrix* (*g' x*))| $*_R$ *f*(*g x*)) *absolutely_integrable_on S*)
**shows** *integral* (*g* ‘ *S*) *f* = *integral S* (λ*x*. |*det* (*matrix* (*g' x*))| $*_R$ *f*(*g x*))

**corollary** *absolutely_integrable_change_of_variables_1*:
**fixes** $f$ :: *real* ⇒ *real*^'*n*::{*finite*,*wellorder*} **and** $g$ :: *real* ⇒ *real*
**assumes** *S*: *S* ∈ *sets lebesgue*
**and** *der_g*: $\bigwedge x.\ x \in S \Longrightarrow$ (*g has_vector_derivative g' x*) (*at x within S*)
**and** *inj*: *inj_on g S*
**shows** (*f absolutely_integrable_on g* ‘ *S* ⟷

$(\lambda x.\ |g'\ x| *_R f(g\ x))$ *absolutely_integrable_on S*)

### 6.41.6 Change of variables for integrals: special case of linear function

### 6.41.7 Change of variable for measure

**end**

## 6.42 Lipschitz Continuity

**theory** *Lipschitz*
**imports**
*Derivative Abstract_Metric_Spaces*
**begin**

**definition** *lipschitz_on*
**where** *lipschitz_on C U f* $\longleftrightarrow$ (*0* $\leq$ *C* $\wedge$ ($\forall$ *x* $\in$ *U*. $\forall$ *y*$\in$*U*. *dist* (*f x*) (*f y*) $\leq$ *C* $*$ *dist x y*))
**notation** *lipschitz_on* (*_−lipschitz'_on* [*1000*])
**proposition** *lipschitz_on_uniformly_continuous*:
**assumes** *L−lipschitz_on X f*
**shows** *uniformly_continuous_on X f*

**proposition** *lipschitz_on_continuous_on*:
*continuous_on X f* **if** *L−lipschitz_on X f*
**proposition** *bounded_derivative_imp_lipschitz*:
**assumes** $\bigwedge x.\ x \in X \Longrightarrow$ (*f has_derivative f′ x*) (*at x within X*)
**assumes** *convex*: *convex X*
**assumes** $\bigwedge x.\ x \in X \Longrightarrow$ *onorm* (*f′ x*) $\leq$ *C 0* $\leq$ *C*
**shows** *C−lipschitz_on X f*

### 6.42.1 Local Lipschitz continuity

**proposition** *lipschitz_on_closed_Union*:
**assumes** $\bigwedge i.\ i \in I \Longrightarrow$ *lipschitz_on M* (*U i*) *f*
$\bigwedge i.\ i \in I \Longrightarrow$ *closed* (*U i*)
*finite I*
*M* $\geq$ *0*
{*u*..(*v*::*real*)} $\subseteq$ ($\bigcup$ *i*$\in$*I*. *U i*)
**shows** *lipschitz_on M* {*u*..*v*} *f*

### 6.42.2 Local Lipschitz continuity (uniform for a family of functions)

**definition** *local_lipschitz*::
$'a$::*metric_space set* ⇒ $'b$::*metric_space set* ⇒ ($'a$ ⇒ $'b$ ⇒ $'c$::*metric_space*) ⇒ *bool*
**where**
*local_lipschitz T X f* ≡ ∀ *x* ∈ *X*. ∀ *t* ∈ *T*.
∃ *u*>*0*. ∃ *L*. ∀ *t* ∈ *cball t u* ∩ *T*. *L*−*lipschitz_on* (*cball x u* ∩ *X*) (*f t*)

**proposition** *c1_implies_local_lipschitz*:
**fixes** *T*::*real set* **and** *X*::$'a$::{*banach*,*heine_borel*} *set*
**and** *f*::*real* ⇒ $'a$ ⇒ $'a$
**assumes** *f'*: ⋀*t x*. *t* ∈ *T* ⟹ *x* ∈ *X* ⟹ (*f t has_derivative blinfun_apply* (*f'* (*t*, *x*))) (*at x*)
**assumes** *cont_f'*: *continuous_on* (*T* × *X*) *f'*
**assumes** *open T*
**assumes** *open X*
**shows** *local_lipschitz T X f*

**end**
**theory**
*Multivariate_Analysis*
**imports**
*Ordered_Euclidean_Space*
*Determinants*
*Cross3*
*Lipschitz*
*Starlike*
**beginend**

## 6.43 Volume of a Simplex

**theory** *Simplex_Content*
**imports** *Change_Of_Vars*
**begin**

**theorem** *content_std_simplex*:
*measure lborel* (*convex hull* (*insert 0 Basis* :: $'a$ :: *euclidean_space set*)) =
*1* / *fact DIM*($'a$)

**proposition** *measure_lebesgue_linear_transformation*:
**fixes** *A* :: (*real* ^ $'n$ :: {*finite*, *wellorder*}) *set*
**fixes** *f* :: _ ⇒ *real* ^ $'n$ :: {*finite*, *wellorder*}
**assumes** *bounded A A* ∈ *sets lebesgue linear f*
**shows** *measure lebesgue* (*f* ‘ *A*) = |*det* (*matrix f*)| ∗ *measure lebesgue A*

**theorem** *content_simplex*:

**fixes** *X* :: (*real* ^ *'n* :: {*finite*, *wellorder*}) *set* **and** *f* :: *'n* :: _ ⇒ *real* ^ (*'n* :: _)
**assumes** *finite X card X* = *Suc CARD*(*'n*) **and** *x0*: *x0* ∈ *X* **and** *bij*: *bij_betw f UNIV* (*X* − {*x0*})
**defines** *M* ≡ (χ *i*. χ *j*. *f j* $ *i* − *x0* $ *i*)
**shows** *content* (*convex hull X*) = |*det M*| / *fact* (*CARD*(*'n*))

**theorem** *content_triangle*:
**fixes** *A B C* :: *real* ^ *2*
**shows** *content* (*convex hull* {*A*, *B*, *C*}) =
|(*C* $ *1* − *A* $ *1*) ∗ (*B* $ *2* − *A* $ *2*) − (*B* $ *1* − *A* $ *1*) ∗ (*C* $ *2* − *A* $ *2*)| / *2*

**theorem** *heron*:
**fixes** *A B C* :: *real* ^ *2*
**defines** *a* ≡ *dist B C* **and** *b* ≡ *dist A C* **and** *c* ≡ *dist A B*
**defines** *s* ≡ (*a* + *b* + *c*) / *2*
**shows** *content* (*convex hull* {*A*, *B*, *C*}) = *sqrt* (*s* ∗ (*s* − *a*) ∗ (*s* − *b*) ∗ (*s* − *c*))

**end**

## 6.44 Convergence of Formal Power Series

**theory** *FPS_Convergence*
**imports**
*Generalised_Binomial_Theorem*
*HOL−Computational_Algebra.Formal_Power_Series*
**begin**

### 6.44.1 Basic properties of convergent power series

**definition** *fps_conv_radius* :: *'a* :: {*banach*, *real_normed_div_algebra*} *fps* ⇒ *ereal* **where**
*fps_conv_radius f* = *conv_radius* (*fps_nth f*)

**definition** *eval_fps* :: *'a* :: {*banach*, *real_normed_div_algebra*} *fps* ⇒ *'a* ⇒ *'a* **where**
*eval_fps f z* = (∑ *n*. *fps_nth f n* ∗ *z* ^ *n*)

**theorem** *sums_eval_fps*:
**fixes** *f* :: *'a* :: {*banach*, *real_normed_div_algebra*} *fps*
**assumes** *norm z* < *fps_conv_radius f*
**shows** (λ*n*. *fps_nth f n* ∗ *z* ^ *n*) *sums eval_fps f z*

### 6.44.2 Evaluating power series

**theorem** *eval_fps_deriv*:
**assumes** *norm z* < *fps_conv_radius f*

**shows** *eval_fps* (*fps_deriv f*) *z* = *deriv* (*eval_fps f*) *z*

**theorem** *fps_nth_conv_deriv*:
**fixes** *f* :: *complex fps*
**assumes** *fps_conv_radius f* > *0*
**shows** *fps_nth f n* = (*deriv* ^^ *n*) (*eval_fps f*) *0* / *fact n*

**theorem** *eval_fps_eqD*:
**fixes** *f g* :: *complex fps*
**assumes** *fps_conv_radius f* > *0* *fps_conv_radius g* > *0*
**assumes** *eventually* (λ*z*. *eval_fps f z* = *eval_fps g z*) (*nhds 0*)
**shows** *f* = *g*

### 6.44.3 Power series expansions of analytic functions

**definition**
*has_fps_expansion* :: (′*a* :: {*banach*,*real_normed_div_algebra*} ⇒ ′*a*) ⇒ ′*a fps* ⇒ *bool*
(**infixl** *has*′*_fps*′*_expansion 60*)
**where** (*f has_fps_expansion F*) ⟷
*fps_conv_radius F* > *0* ∧ *eventually* (λ*z*. *eval_fps F z* = *f z*) (*nhds 0*)

**end**
**theory** *Smooth_Paths*
**imports**
*Retracts*
**begin**

### 6.44.4 Piecewise differentiability of paths

### 6.44.5 Valid paths, and their start and finish

**definition** *valid_path* :: (*real* ⇒ ′*a* :: *real_normed_vector*) ⇒ *bool*
**where** *valid_path f* ≡ *f piecewise_C1_differentiable_on* {*0*..*1*::*real*}

**end**

## 6.45 Metrics on product spaces

**theory** *Function_Metric*
**imports**
*Function_Topology*
*Elementary_Metric_Spaces*
**begininstantiation** *fun* :: (*countable*, *metric_space*) *metric_space*
**begin**

**definition** *dist_fun_def*:
*dist x y* = (∑ *n*. (*1*/*2*)^*n* ∗ *min* (*dist* (*x* (*from_nat n*)) (*y* (*from_nat n*))) *1*)

**definition** *uniformity_fun_def*:
(*uniformity*::((′*a* ⇒ ′*b*) × (′*a* ⇒ ′*b*)) *filter*) = (*INF e*∈{*0*<..}. *principal* {(*x*, *y*). *dist* (*x*::(′*a*⇒′*b*)) *y* < *e*})
**end**
**theory** *Analysis*
**imports**

*Convex*
*Determinants*

*FSigma*
*Sum_Topology*
*Abstract_Topological_Spaces*
*Abstract_Metric_Spaces*
*Urysohn*
*Connected*
*Abstract_Limits*
*Isolated*

*Elementary_Normed_Spaces*
*Norm_Arith*

*Convex_Euclidean_Space*
*Operator_Norm*

*Line_Segment*
*Derivative*
*Cartesian_Euclidean_Space*
*Weierstrass_Theorems*

*Ball_Volume*
*Integral_Test*
*Improper_Integral*
*Equivalence_Measurable_On_Borel*
*Lebesgue_Integral_Substitution*
*Embed_Measure*
*Complete_Measure*
*Radon_Nikodym*
*Fashoda_Theorem*
*Cross3*
*Homeomorphism*
*Bounded_Continuous_Function*
*Abstract_Topology*
*Product_Topology*
*Lindelof_Spaces*
*Infinite_Products*

*Infinite_Sum*
*Infinite_Set_Sum*
*Polytope*
*Jordan_Curve*
*Poly_Roots*
*Generalised_Binomial_Theorem*
*Gamma_Function*
*Change_Of_Vars*
*Multivariate_Analysis*
*Simplex_Content*
*FPS_Convergence*
*Smooth_Paths*
*Abstract_Euclidean_Space*
*Function_Metric*
**begin**

**end**

# Bibliography


[1]