Complex Analysis

September 11, 2023

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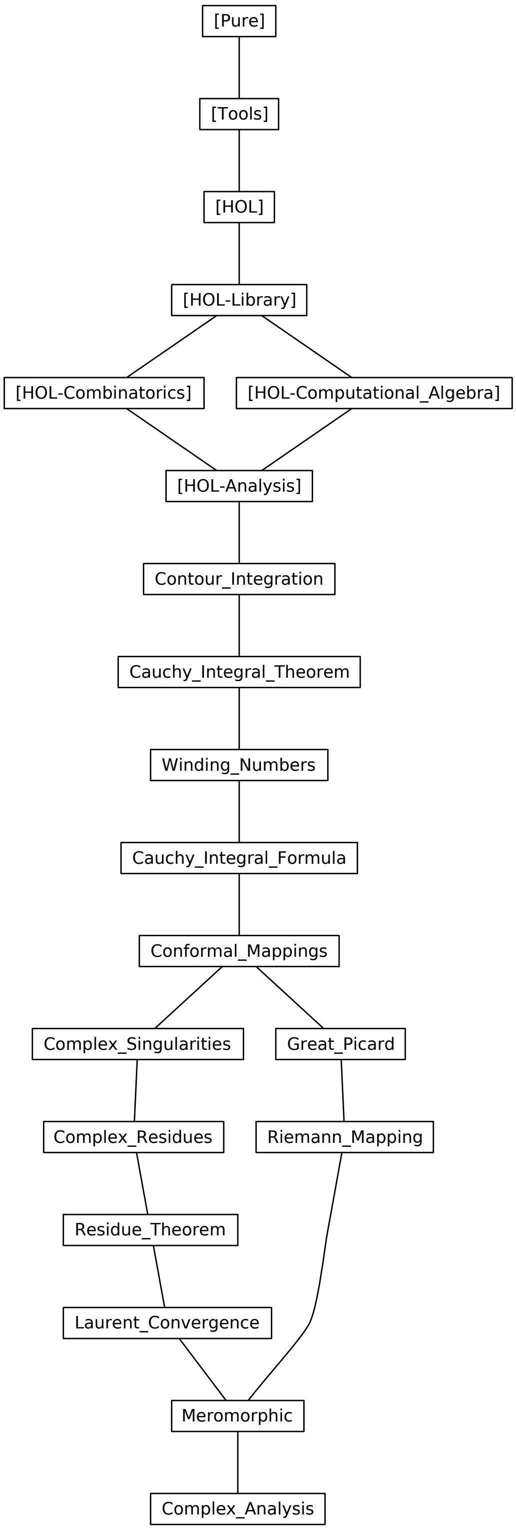
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1 Contour integration

theory *Contour_Integration*
imports *HOL−Analysis.Analysis*
begin

1.1 Definition

definition *has_contour_integral* :: (*complex* ⇒ *complex*) ⇒ *complex* ⇒ (*real* ⇒ *complex*) ⇒ *bool*
(**infixr** *has′_contour′_integral 50*)
where (*f has_contour_integral i*) *g* ≡
((λ*x*. *f*(*g x*) ∗ *vector_derivative g* (*at x within* {*0*..*1*}))
has_integral i) {*0*..*1*}

definition *contour_integrable_on*
(**infixr** *contour′_integrable′_on 50*)
where *f contour_integrable_on g* ≡ ∃ *i*. (*f has_contour_integral i*) *g*

definition *contour_integral*
where *contour_integral g f* ≡ *SOME i*. (*f has_contour_integral i*) *g* ∨ ¬ *f contour_integrable_on g* ∧ *i*=*0*

1.2 Relation to subpath construction

1.3 Cauchy's theorem where there's a primitive

corollary *Cauchy_theorem_primitive*:
assumes ⋀*x*. *x* ∈ *S* ⟹ (*f has_field_derivative f′ x*) (*at x within S*)
and *valid_path g* *path_image g* ⊆ *S pathfinish g* = *pathstart g*
shows (*f′ has_contour_integral 0*) *g*

1.4 Reversing the order in a double path integral

proposition *contour_integral_swap*:
assumes *fcon*: *continuous_on* (*path_image g* × *path_image h*) (λ(*y1*,*y2*). *f y1 y2*)
and *vp*: *valid_path g valid_path h*
and *gvcon*: *continuous_on* {*0*..*1*} (λ*t*. *vector_derivative g* (*at t*))
and *hvcon*: *continuous_on* {*0*..*1*} (λ*t*. *vector_derivative h* (*at t*))
shows *contour_integral g* (λ*w*. *contour_integral h* (*f w*)) =
contour_integral h (λ*z*. *contour_integral g* (λ*w*. *f w z*))

1.5 Partial circle path

definition *part_circlepath* :: [*complex*, *real*, *real*, *real*, *real*] ⇒ *complex*
where *part_circlepath z r s t* ≡ λ*x*. *z* + *of_real r* ∗ *exp* (i ∗ *of_real* (*linepath s t x*))

proposition *path_image_part_circlepath*:
assumes $s \le t$
shows *path_image* (*part_circlepath z r s t*) = {*z* + *r* ∗ *exp*(i ∗ *of_real x*) | *x*. $s \le x \wedge x \le t$}

corollary *contour_integral_bound_part_circlepath_strong*:
assumes *f contour_integrable_on part_circlepath z r s t*
and *finite k* **and** $0 \le B$ $0 < r$ $s \le t$
and ⋀*x*. *x* ∈ *path_image*(*part_circlepath z r s t*) − *k* ⟹ *norm*(*f x*) ≤ *B*
shows *cmod* (*contour_integral* (*part_circlepath z r s t*) *f*) ≤ *B* ∗ *r* ∗ (*t* − *s*)

1.6 Special case of one complete circle

definition *circlepath* :: [*complex*, *real*, *real*] ⇒ *complex*
where *circlepath z r* ≡ *part_circlepath z r 0* (*2*∗*pi*)

1.7 Uniform convergence of path integral

proposition *contour_integral_uniform_limit*:
assumes *ev_fint*: *eventually* (λ*n*::′*a*. (*f n*) *contour_integrable_on* γ) *F*
and *ul_f*: *uniform_limit* (*path_image* γ) *f l F*
and *noleB*: ⋀*t*. *t* ∈ {*0*..*1*} ⟹ *norm* (*vector_derivative* γ (*at t*)) ≤ *B*
and γ: *valid_path* γ
and [*simp*]: ¬ *trivial_limit F*
shows *l contour_integrable_on* γ ((λ*n*. *contour_integral* γ (*f n*)) ⟶ *contour_integral* γ *l*) *F*

end

2 Complex Path Integrals and Cauchy's Integral Theorem

theory *Cauchy_Integral_Theorem*
imports
HOL−Analysis.Analysis
Contour_Integration
begin

proposition *Cauchy_theorem_triangle_interior*:
assumes *contf*: *continuous_on* (*convex hull* {*a*,*b*,*c*}) *f*
and *holf*: *f holomorphic_on interior* (*convex hull* {*a*,*b*,*c*})
shows (*f has_contour_integral 0*) (*linepath a b* +++ *linepath b c* +++ *linepath c a*)

2.1 Cauchy's theorem for a convex set

corollary *Cauchy_theorem_convex_simple*:
assumes *holf*: *f holomorphic_on S*
and *convex S valid_path g path_image g* ⊆ *S pathfinish g* = *pathstart g*
shows (*f has_contour_integral 0*) *g*

2.2 Homotopy forms of Cauchy's theorem

proposition *Cauchy_theorem_homotopic_paths*:
assumes *hom*: *homotopic_paths S g h*
and *open S* **and** *f*: *f holomorphic_on S*
and *vpg*: *valid_path g* **and** *vph*: *valid_path h*
shows *contour_integral g f* = *contour_integral h f*

proposition *Cauchy_theorem_homotopic_loops*:
assumes *hom*: *homotopic_loops S g h*
and *open S* **and** *f*: *f holomorphic_on S*
and *vpg*: *valid_path g* **and** *vph*: *valid_path h*
shows *contour_integral g f* = *contour_integral h f*

end

3 Winding numbers

theory *Winding_Numbers*
imports *Cauchy_Integral_Theorem*
begin

3.1 Definition

definition *winding_number_prop* :: [*real* ⇒ *complex*, *complex*, *real*, *real* ⇒ *complex*, *complex*] ⇒ *bool* **where**
winding_number_prop γ *z e p n* ≡
valid_path p ∧ *z* ∉ *path_image p* ∧
pathstart p = *pathstart* γ ∧
pathfinish p = *pathfinish* γ ∧

$(\forall t \in \{0..1\}.\ norm(\gamma\ t - p\ t) < e) \wedge$
$contour\_integral\ p\ (\lambda w.\ 1/(w - z)) = 2 * pi * \mathrm{i} * n$

definition *winding_number*:: [*real* ⇒ *complex*, *complex*] ⇒ *complex* **where**
$winding\_number\ \gamma\ z \equiv SOME\ n.\ \forall e > 0.\ \exists p.\ winding\_number\_prop\ \gamma\ z\ e\ p\ n$

proposition *winding_number_valid_path*:
assumes $valid\_path\ \gamma\ z \notin path\_image\ \gamma$
shows $winding\_number\ \gamma\ z = 1/(2*pi*\mathrm{i}) * contour\_integral\ \gamma\ (\lambda w.\ 1/(w - z))$

proposition *has_contour_integral_winding_number*:
assumes γ: $valid\_path\ \gamma\ z \notin path\_image\ \gamma$
shows $((\lambda w.\ 1/(w - z))\ has\_contour\_integral\ (2*pi*\mathrm{i}*winding\_number\ \gamma\ z))\ \gamma$

3.2 The winding number is an integer

theorem *integer_winding_number*:
$[\![path\ \gamma;\ pathfinish\ \gamma = pathstart\ \gamma;\ z \notin path\_image\ \gamma]\!] \Longrightarrow winding\_number\ \gamma\ z \in \mathbb{Z}$

3.3 Continuity of winding number and invariance on connected sets

theorem *continuous_at_winding_number*:
fixes *z*::*complex*
assumes γ: $path\ \gamma$ **and** *z*: $z \notin path\_image\ \gamma$
shows $continuous\ (at\ z)\ (winding\_number\ \gamma)$

corollary *continuous_on_winding_number*:
$path\ \gamma \Longrightarrow continuous\_on\ (- path\_image\ \gamma)\ (\lambda w.\ winding\_number\ \gamma\ w)$

3.4 Winding number is zero "outside" a curve

proposition *winding_number_zero_in_outside*:
assumes γ: $path\ \gamma$ **and** *loop*: $pathfinish\ \gamma = pathstart\ \gamma$ **and** *z*: $z \in outside\ (path\_image\ \gamma)$
shows $winding\_number\ \gamma\ z = 0$

proposition *winding_number_part_circlepath_pos_less*:
assumes $s < t$ **and** *no*: $norm(w - z) < r$
shows $0 < Re\ (winding\_number(part\_circlepath\ z\ r\ s\ t)\ w)$

proposition *winding_number_circlepath*:

assumes *norm*$(w - z) < r$ **shows** *winding_number*(*circlepath z r*) $w = 1$

3.5 Winding number for a triangle

proposition *winding_number_triangle*:
assumes *z*: $z \in$ *interior*(*convex hull* $\{a,b,c\}$)
shows *winding_number*(*linepath a b* +++ *linepath b c* +++ *linepath c a*) $z =$
(*if* $0 <$ *Im*($(b - a) *$ *cnj* $(b - z)$) *then* 1 *else* -1)

3.6 Winding numbers for simple closed paths

proposition *simple_closed_path_winding_number_inside*:
assumes *simple_path* γ
obtains $\bigwedge z.\ z \in$ *inside*(*path_image* γ) $\Longrightarrow$ *winding_number* $\gamma\ z = 1$
| $\bigwedge z.\ z \in$ *inside*(*path_image* γ) $\Longrightarrow$ *winding_number* $\gamma\ z = -1$

3.7 Winding number for rectangular paths

proposition *winding_number_rectpath*:
assumes $z \in$ *box a1 a3*
shows *winding_number* (*rectpath a1 a3*) $z = 1$

proposition *winding_number_rectpath_outside*:
assumes *Re a1* $\leq$ *Re a3 Im a1* $\leq$ *Im a3*
assumes $z \notin$ *cbox a1 a3*
shows *winding_number* (*rectpath a1 a3*) $z = 0$

end

4 Cauchy's Integral Formula

theory *Cauchy_Integral_Formula*
imports *Winding_Numbers*
begin

4.1 Proof

theorem *Cauchy_integral_formula_convex_simple*:
assumes *convex S* **and** *holf*: *f holomorphic_on S* **and** $z \in$ *interior S valid_path* γ *path_image* $\gamma \subseteq S - \{z\}$
pathfinish γ = *pathstart* γ
shows $((\lambda w.\ f\ w\ /\ (w - z))$ *has_contour_integral* $(2*pi * \mathrm{i} *$ *winding_number* $\gamma\ z * f\ z))\ \gamma$

theorem *Cauchy_integral_circlepath*:
assumes *contf*: *continuous_on* (*cball z r*) *f* **and** *holf*: *f holomorphic_on* (*ball z r*) **and** *wz*: *norm*($w - z$) $<$ *r*
shows (($\lambda u.$ *f u*/($u - w$)) *has_contour_integral* ($2 * $ *of_real pi* $*$ i $*$ *f w*)) (*circlepath z r*)

4.2 Existence of all higher derivatives

proposition *derivative_is_holomorphic*:
assumes *open S*
and *fder*: $\bigwedge z.\ z \in S \Longrightarrow$ (*f has_field_derivative f′ z*) (*at z*)
shows *f′ holomorphic_on S*

4.3 Morera's theorem

proposition *Morera_triangle*:
⟦*continuous_on S f*; *open S*;
$\bigwedge a\ b\ c.$ *convex hull* $\{a,b,c\} \subseteq S$
$\longrightarrow$ *contour_integral* (*linepath a b*) *f* +
contour_integral (*linepath b c*) *f* +
contour_integral (*linepath c a*) *f* = *0*⟧
$\Longrightarrow$ *f analytic_on S*

4.4 Combining theorems for higher derivatives including Leibniz rule

proposition *no_isolated_singularity*:
fixes *z*::*complex*
assumes *f*: *continuous_on S f* **and** *holf*: *f holomorphic_on* ($S - K$) **and** *S*: *open S* **and** *K*: *finite K*
shows *f holomorphic_on S*

proposition *Cauchy_integral_formula_convex*:
assumes *S*: *convex S* **and** *K*: *finite K* **and** *contf*: *continuous_on S f*
and *fcd*: ($\bigwedge x.\ x \in$ *interior* $S - K \Longrightarrow$ *f field_differentiable at x*)
and *z*: $z \in$ *interior S* **and** *vpg*: *valid_path* γ
and *pasz*: *path_image* $\gamma \subseteq S - \{z\}$ **and** *loop*: *pathfinish* γ = *pathstart* γ
shows (($\lambda w.$ *f w* / ($w - z$)) *has_contour_integral* (*2*$*$*pi* $*$ i $*$ *winding_number* γ $z * f z$)) γ

corollary *Cauchy_contour_integral_circlepath*:
assumes *continuous_on* (*cball z r*) *f f holomorphic_on ball z r w* $\in$ *ball z r*
shows *contour_integral*(*circlepath z r*) ($\lambda u.$ *f u*/($u - w$)^(*Suc k*)) = ($2 * $ *pi* $*$ i) $*$ (*deriv* ^^ *k*) *f w* / (*fact k*)

4.5 A holomorphic function is analytic, i.e. has local power series

theorem *holomorphic_power_series*:
  assumes *holf*: *f holomorphic_on ball z r*
    and *w*: *w* ∈ *ball z r*
  shows ((λ*n*. (*deriv* ^^ *n*) *f z* / (*fact n*) ∗ (*w* − *z*)^*n*) *sums f w*)

4.6 The Liouville theorem and the Fundamental Theorem of Algebra

proposition *Liouville_weak*:
  assumes *f holomorphic_on UNIV* **and** (*f* ⟶ *l*) *at_infinity*
  shows *f z* = *l*

proposition *Liouville_weak_inverse*:
  assumes *f holomorphic_on UNIV* **and** *unbounded*: ⋀*B*. *eventually* (λ*x*. *norm* (*f x*) ≥ *B*) *at_infinity*
  obtains *z* **where** *f z* = *0*

theorem *fundamental_theorem_of_algebra*:
  fixes *a* :: *nat* ⇒ *complex*
  assumes *a 0* = *0* ∨ (∃ *i* ∈ {*1*..*n*}. *a i* ≠ *0*)
  obtains *z* **where** (∑ *i*≤*n*. *a i* ∗ *z*^*i*) = *0*

4.7 Weierstrass convergence theorem

proposition *has_complex_derivative_uniform_limit*:
  fixes *z*::*complex*
  assumes *cont*: *eventually* (λ*n*. *continuous_on* (*cball z r*) (*f n*) ∧
    (∀ *w* ∈ *ball z r*. ((*f n*) *has_field_derivative* (*f′ n w*)) (*at w*))) *F*
    and *ulim*: *uniform_limit* (*cball z r*) *f g F*
    and *F*: ¬ *trivial_limit F* **and** *0* < *r*
  obtains *g′* **where**
    continuous_on (*cball z r*) *g*
    ⋀*w*. *w* ∈ *ball z r* ⟹ (*g has_field_derivative* (*g′ w*)) (*at w*) ∧ ((λ*n*. *f′ n w*) ⟶ *g′ w*) *F*

4.8 On analytic functions defined by a series

corollary *holomorphic_iff_power_series*:
  f holomorphic_on ball z r ⟷
    (∀ *w* ∈ *ball z r*. (λ*n*. (*deriv* ^^ *n*) *f z* / (*fact n*) ∗ (*w* − *z*)^*n*) *sums f w*)

4.9 General, homology form of Cauchy's theorem

theorem *Cauchy_integral_formula_global*:
assumes *S*: *open S* **and** *holf*: *f holomorphic_on S*
and *z*: $z \in S$ **and** *vpg*: *valid_path* γ
and *pasz*: *path_image* $\gamma \subseteq S - \{z\}$ **and** *loop*: *pathfinish* γ = *pathstart* γ
and *zero*: $\bigwedge w.\ w \notin S \Longrightarrow$ *winding_number* γ *w* = *0*
shows $((\lambda w.\ f\ w\ /\ (w - z))$ *has_contour_integral* (*2*∗*pi* ∗ i ∗ *winding_number* γ *z* ∗ *f z*)) γ

theorem *Cauchy_theorem_global*:
assumes *S*: *open S* **and** *holf*: *f holomorphic_on S*
and *vpg*: *valid_path* γ **and** *loop*: *pathfinish* γ = *pathstart* γ
and *pas*: *path_image* $\gamma \subseteq S$
and *zero*: $\bigwedge w.\ w \notin S \Longrightarrow$ *winding_number* γ *w* = *0*
shows (*f has_contour_integral 0*) γ

corollary *Cauchy_theorem_global_outside*:
assumes *open S f holomorphic_on S valid_path* γ *pathfinish* γ = *pathstart* γ *path_image* $\gamma \subseteq S$
$\bigwedge w.\ w \notin S \Longrightarrow w \in$ *outside*(*path_image* γ)
shows (*f has_contour_integral 0*) γ

4.10 Cauchy's inequality and more versions of Liouville

theorem *Liouville_theorem*:
assumes *holf*: *f holomorphic_on UNIV*
and *bf*: *bounded* (*range f*)
shows *f constant_on UNIV*

4.11 Complex functions and power series

definition *fps_expansion* :: (*complex* ⇒ *complex*) ⇒ *complex* ⇒ *complex fps*
where
fps_expansion f z0 = *Abs_fps* (λ*n*. (*deriv* ^^ *n*) *f z0* / *fact n*)

end

5 Conformal Mappings and Consequences of Cauchy's Integral Theorem

theory *Conformal_Mappings*
imports *Cauchy_Integral_Formula*

begin

5.1 Analytic continuation

proposition *isolated_zeros*:
assumes *holf*: f *holomorphic_on* S
and *open* S *connected* S $\xi \in S$ $f\ \xi = 0$ $\beta \in S$ $f\ \beta \neq 0$
obtains r **where** $0 < r$ **and** *ball* $\xi\ r \subseteq S$ **and**
$\bigwedge z.\ z \in ball\ \xi\ r - \{\xi\} \Longrightarrow f\ z \neq 0$

proposition *analytic_continuation*:
assumes *holf*: f *holomorphic_on* S
and *open* S **and** *connected* S
and $U \subseteq S$ **and** $\xi \in S$
and ξ *islimpt* U
and *fU0* [*simp*]: $\bigwedge z.\ z \in U \Longrightarrow f\ z = 0$
and $w \in S$
shows $f\ w = 0$

corollary *analytic_continuation_open*:
assumes *open* s **and** *open* s' **and** $s \neq \{\}$ **and** *connected* s'
and $s \subseteq s'$
assumes f *holomorphic_on* s' **and** g *holomorphic_on* s'
and $\bigwedge z.\ z \in s \Longrightarrow f\ z = g\ z$
assumes $z \in s'$
shows $f\ z = g\ z$

corollary *analytic_continuation'*:
assumes f *holomorphic_on* S *open* S *connected* S
and $U \subseteq S$ $\xi \in S$ ξ *islimpt* U
and f *constant_on* U
shows f *constant_on* S

5.2 Open mapping theorem

theorem *open_mapping_thm*:
assumes *holf*: f *holomorphic_on* S
and S: *open* S **and** *connected* S
and *open* U **and** $U \subseteq S$
and *fne*: $\neg$ f *constant_on* S
shows *open* $(f\ `\ U)$

5.3 Maximum modulus principle

proposition *maximum_modulus_principle*:
assumes *holf*: f *holomorphic_on* S
and S: *open* S **and** *connected* S
and *open* U **and** $U \subseteq S$ **and** $\xi \in U$
and *no*: $\bigwedge z.\ z \in U \Longrightarrow norm(f\ z) \leq norm(f\ \xi)$

shows *f constant_on S*

proposition *maximum_modulus_frontier*:
assumes *holf*: *f holomorphic_on* (*interior S*)
and *contf*: *continuous_on* (*closure S*) *f*
and *bos*: *bounded S*
and *leB*: $\bigwedge z.\ z \in \mathit{frontier}\ S \Longrightarrow \mathit{norm}(f\ z) \le B$
and $\xi \in S$
shows $\mathit{norm}(f\ \xi) \le B$

5.4 Relating invertibility and nonvanishing of derivative

proposition *holomorphic_has_inverse*:
assumes *holf*: *f holomorphic_on S*
and *open S* **and** *injf*: *inj_on f S*
obtains *g* **where** *g holomorphic_on* (*f* ‘ *S*)
$\bigwedge z.\ z \in S \Longrightarrow \mathit{deriv}\ f\ z * \mathit{deriv}\ g\ (f\ z) = 1$
$\bigwedge z.\ z \in S \Longrightarrow g(f\ z) = z$

5.5 The Schwarz Lemma

proposition *Schwarz_Lemma*:
assumes *holf*: *f holomorphic_on* (*ball 0 1*) **and** [*simp*]: $f\ 0 = 0$
and *no*: $\bigwedge z.\ \mathit{norm}\ z < 1 \Longrightarrow \mathit{norm}\ (f\ z) < 1$
and ξ: $\mathit{norm}\ \xi < 1$
shows $\mathit{norm}\ (f\ \xi) \le \mathit{norm}\ \xi$ **and** $\mathit{norm}(\mathit{deriv}\ f\ 0) \le 1$
and $((\exists z.\ \mathit{norm}\ z < 1 \wedge z \neq 0 \wedge \mathit{norm}(f\ z) = \mathit{norm}\ z)$
$\vee\ \mathit{norm}(\mathit{deriv}\ f\ 0) = 1)$
$\Longrightarrow \exists \alpha.\ (\forall z.\ \mathit{norm}\ z < 1 \longrightarrow f\ z = \alpha * z) \wedge \mathit{norm}\ \alpha = 1$
(**is** *?P* $\Longrightarrow$ *?Q*)

corollary *Schwarz_Lemma′*:
assumes *holf*: *f holomorphic_on* (*ball 0 1*) **and** [*simp*]: $f\ 0 = 0$
and *no*: $\bigwedge z.\ \mathit{norm}\ z < 1 \Longrightarrow \mathit{norm}\ (f\ z) < 1$
shows $((\forall \xi.\ \mathit{norm}\ \xi < 1 \longrightarrow \mathit{norm}\ (f\ \xi) \le \mathit{norm}\ \xi)$
$\wedge\ \mathit{norm}(\mathit{deriv}\ f\ 0) \le 1)$
$\wedge\ (((\exists z.\ \mathit{norm}\ z < 1 \wedge z \neq 0 \wedge \mathit{norm}(f\ z) = \mathit{norm}\ z)$
$\vee\ \mathit{norm}(\mathit{deriv}\ f\ 0) = 1)$
$\longrightarrow (\exists \alpha.\ (\forall z.\ \mathit{norm}\ z < 1 \longrightarrow f\ z = \alpha * z) \wedge \mathit{norm}\ \alpha = 1))$

5.6 The Schwarz reflection principle

proposition *Schwarz_reflection*:
  assumes *open S* **and** *cnjs*: *cnj ‘ S ⊆ S*
    and *holf*: *f holomorphic_on* (*S* ∩ {*z*. *0* < *Im z*})
    and *contf*: *continuous_on* (*S* ∩ {*z*. *0* ≤ *Im z*}) *f*
    and *f*: ⋀*z*. ⟦*z* ∈ *S*; *z* ∈ ℝ⟧ ⟹ (*f z*) ∈ ℝ
   shows (λ*z*. *if 0* ≤ *Im z then f z else cnj*(*f*(*cnj z*))) *holomorphic_on S*

5.7 Bloch's theorem

proposition *Bloch_unit*:
  assumes *holf*: *f holomorphic_on ball a 1* **and** [*simp*]: *deriv f a* = *1*
  obtains *b r* **where** *1*/*12* < *r* **and** *ball b r* ⊆ *f* ‘ (*ball a 1*)

theorem *Bloch*:
  assumes *holf*: *f holomorphic_on ball a r* **and** *0* < *r*
    and *r'*: *r'* ≤ *r* ∗ *norm* (*deriv f a*) / *12*
  obtains *b* **where** *ball b r'* ⊆ *f* ‘ (*ball a r*)

corollary *Bloch_general*:
  assumes *holf*: *f holomorphic_on S* **and** *a* ∈ *S*
    and *tle*: ⋀*z*. *z* ∈ *frontier S* ⟹ *t* ≤ *dist a z*
    and *rle*: *r* ≤ *t* ∗ *norm*(*deriv f a*) / *12*
  obtains *b* **where** *ball b r* ⊆ *f* ‘ *S*

end
theory *Complex_Singularities*
  imports *Conformal_Mappings*
begin

5.8 Non-essential singular points

definition *is_pole* ::
  (*'a*::*topological_space* ⇒ *'b*::*real_normed_vector*) ⇒ *'a* ⇒ *bool* **where**
  is_pole f a = (*LIM x* (*at a*). *f x* :> *at_infinity*)

5.9 The order of non-essential singularities (i.e. removable singularities or poles)

definition *zorder* :: (*complex* ⇒ *complex*) ⇒ *complex* ⇒ *int* **where**
  zorder f z = (*THE n*. (∃ *h r*. *r*>*0* ∧ *h holomorphic_on cball z r* ∧ *h z*≠*0*
        ∧ (∀ *w*∈*cball z r* − {*z*}. *f w* = *h w* ∗ (*w*−*z*) *powi n*
        ∧ *h w* ≠*0*)))

definition *zor_poly*

```
  ::[complex ⇒ complex, complex] ⇒ complex ⇒ complex where
 zor_poly f z = (SOME h. ∃r. r > 0 ∧ h holomorphic_on cball z r ∧ h z ≠ 0
                ∧ (∀w∈cball z r − {z}. f w =  h w ∗ (w − z) powi (zorder f z)
                ∧ h w ≠0))
```

5.10 Isolated zeroes

5.11 Isolated points

```
end
theory Complex_Residues
 imports Complex_Singularities
begin
```

5.12 Definition of residues

```
definition residue :: (complex ⇒ complex) ⇒ complex ⇒ complex where
 residue f z = (SOME int. ∃e>0. ∀ε>0. ε<e
   ⟶ (f has_contour_integral 2∗pi∗ i ∗int) (circlepath z ε))

theorem residue_fps_expansion_over_power_at_0:
 assumes f has_fps_expansion F
 shows  residue (λz. f z / z ^ Suc n) 0 = fps_nth F n
```

5.13 Poles and residues of some well-known functions

```
end
```

6 The Residue Theorem, the Argument Principle and Rouché's Theorem

```
theory Residue_Theorem
 imports Complex_Residues HOL−Library.Landau_Symbols
begin
```

6.1 Cauchy's residue theorem

```
theorem Residue_theorem:
 fixes s pts::complex set and f::complex ⇒ complex
  and g::real ⇒ complex
 assumes open s connected s finite pts and
     holo:f holomorphic_on s−pts and
```

valid_path g **and**
loop:*pathfinish g* = *pathstart g* **and**
path_image g ⊆ *s*−*pts* **and**
homo:∀ *z*. (*z* ∉ *s*) ⟶ *winding_number g z* = *0*
shows *contour_integral g f* = *2* ∗ *pi* ∗ i ∗(∑ *p*∈*pts*. *winding_number g p* ∗ *residue f p*)

6.2 The argument principle

theorem *argument_principle*:
fixes *f*::*complex* ⇒ *complex* **and** *poles s*:: *complex set*
defines *pz* ≡ {*w*∈*s*. *f w* = *0* ∨ *w* ∈ *poles*} — *pz* is the set of poles and zeros
assumes *open s connected s* **and**
f_holo:*f holomorphic_on s*−*poles* **and**
h_holo:*h holomorphic_on s* **and**
valid_path g **and**
loop:*pathfinish g* = *pathstart g* **and**
path_img:*path_image g* ⊆ *s* − *pz* **and**
homo:∀ *z*. (*z* ∉ *s*) ⟶ *winding_number g z* = *0* **and**
finite:*finite pz* **and**
poles:∀ *p*∈*s*∩*poles*. *is_pole f p*
shows *contour_integral g* (λ*x*. *deriv f x* ∗ *h x* / *f x*) = *2* ∗ *pi* ∗ i ∗
(∑ *p*∈*pz*. *winding_number g p* ∗ *h p* ∗ *zorder f p*)
(**is** *?L*=*?R*)

6.3 Coefficient asymptotics for generating functions

theorem
fixes *f* :: *complex* ⇒ *complex* **and** *n* :: *nat* **and** *r* :: *real*
defines *g* ≡ (λ*w*. *f w* / *w* ^ *Suc n*) **and** γ ≡ *circlepath 0 r*
assumes *open A connected A cball 0 r* ⊆ *A r* > *0*
assumes *f holomorphic_on A* − *S S* ⊆ *ball 0 r finite S 0* ∉ *S*
shows *fps_coeff_conv_residues*:
(*deriv* ^^ *n*) *f 0* / *fact n* =
contour_integral γ *g* / (*2* ∗ *pi* ∗ i) − (∑ *z*∈*S*. *residue g z*) (**is** *?thesis1*)
and *fps_coeff_residues_bound*:
(⋀*z*. *norm z* = *r* ⟹ *z* ∉ *k* ⟹ *norm* (*f z*) ≤ *C*) ⟹ *C* ≥ *0* ⟹ *finite k* ⟹
norm ((*deriv* ^^ *n*) *f 0* / *fact n* + (∑ *z*∈*S*. *residue g z*)) ≤ *C* / *r* ^ *n*
corollary *fps_coeff_residues_bigo*:
fixes *f* :: *complex* ⇒ *complex* **and** *r* :: *real*
assumes *open A connected A cball 0 r* ⊆ *A r* > *0*
assumes *f holomorphic_on A* − *S S* ⊆ *ball 0 r finite S 0* ∉ *S*
assumes *g*: *eventually* (λ*n*. *g n* = −(∑ *z*∈*S*. *residue* (λ*z*. *f z* / *z* ^ *Suc n*) *z*)) *sequentially*
(**is** *eventually* (λ*n*. _ = −*?g' n*) _)
shows (λ*n*. (*deriv* ^^ *n*) *f 0* / *fact n* − *g n*) ∈ *O*(λ*n*. *1* / *r* ^ *n*) (**is** (λ*n*. *?c n* − _) ∈ *O*(_))

corollary *fps_coeff_residues_bigo'*:
fixes *f* :: *complex* ⇒ *complex* **and** *r* :: *real*
assumes *exp*: *f has_fps_expansion F*
assumes *open A connected A cball 0 r* ⊆ *A r* > *0*
assumes *f holomorphic_on A* − *S S* ⊆ *ball 0 r finite S 0* ∉ *S*
assumes *eventually* (λ*n*. *g n* = −(∑*z*∈*S*. *residue* (λ*z*. *f z* / *z* ^ *Suc n*) *z*)) *sequentially*
(**is** *eventually* (λ*n*. _ = −*?g'* *n*) _)
shows (λ*n*. *fps_nth F n* − *g n*) ∈ *O*(λ*n*. *1* / *r* ^ *n*) (**is** (λ*n*. *?c n* − _) ∈ *O*(_))

6.4 Rouche's theorem

theorem *Rouche_theorem*:
fixes *f g*::*complex* ⇒ *complex* **and** *s*:: *complex set*
defines *fg*≡(λ*p*. *f p* + *g p*)
defines *zeros_fg*≡{*p*∈*s*. *fg p* = *0*} **and** *zeros_f*≡{*p*∈*s*. *f p* = *0*}
assumes
open s **and** *connected s* **and**
finite zeros_fg **and**
finite zeros_f **and**
f_holo:*f holomorphic_on s* **and**
g_holo:*g holomorphic_on s* **and**
valid_path γ **and**
loop:*pathfinish* γ = *pathstart* γ **and**
path_img:*path_image* γ ⊆ *s* **and**
path_less:∀ *z*∈*path_image* γ. *cmod*(*f z*) > *cmod*(*g z*) **and**
homo:∀ *z*. (*z* ∉ *s*) ⟶ *winding_number* γ *z* = *0*
shows (∑*p*∈*zeros_fg*. *winding_number* γ *p* ∗ *zorder fg p*)
= (∑*p*∈*zeros_f*. *winding_number* γ *p* ∗ *zorder f p*)

end
theory *Laurent_Convergence*
imports *HOL−Computational_Algebra.Formal_Laurent_Series HOL−Library.Landau_Symbols Residue_Theorem*

begin

definition *fls_conv_radius* :: *complex fls* ⇒ *ereal* **where**
fls_conv_radius f = *fps_conv_radius* (*fls_regpart f*)

definition *eval_fls* :: *complex fls* ⇒ *complex* ⇒ *complex* **where**
eval_fls F z = *eval_fps* (*fls_base_factor_to_fps F*) *z* ∗ *z powi fls_subdegree F*

definition
has_laurent_expansion :: (*complex* ⇒ *complex*) ⇒ *complex fls* ⇒ *bool*
(**infixl** *has'_laurent'_expansion 60*)

where $(f\ has\_laurent\_expansion\ F) \longleftrightarrow$
$fls\_conv\_radius\ F > 0 \wedge eventually\ (\lambda z.\ eval\_fls\ F\ z = f\ z)\ (at\ 0)$

theorem *sums_eval_fls*:
fixes *f*
defines $n \equiv fls\_subdegree\ f$
assumes $norm\ z < fls\_conv\_radius\ f$ **and** $z \neq 0 \vee n \geq 0$
shows $(\lambda k.\ fls\_nth\ f\ (int\ k + n) * z\ powi\ (int\ k + n))\ sums\ eval\_fls\ f\ z$

theorem *not_essential_has_laurent_expansion_0*:
assumes *isolated_singularity_at f 0 not_essential f 0*
shows *f has_laurent_expansion laurent_expansion f 0*

end

7 The Great Picard Theorem and its Applications

theory *Great_Picard*
imports *Conformal_Mappings*
begin

7.1 Schottky's theorem

theorem *Schottky*:
assumes *holf*: *f holomorphic_on cball 0 1*
and *nof0*: $norm(f\ 0) \leq r$
and *not01*: $\bigwedge z.\ z \in cball\ 0\ 1 \Longrightarrow \neg(f\ z = 0 \vee f\ z = 1)$
and $0 < t$ $t < 1$ $norm\ z \leq t$
shows $norm(f\ z) \leq exp(pi * exp(pi * (2 + 2 * r + 12 * t / (1 - t))))$

7.2 The Little Picard Theorem

theorem *Landau_Picard*:
obtains *R*

where $\bigwedge z.\ 0 < R\ z$
$\bigwedge f.\ [\![f\ holomorphic\_on\ cball\ 0\ (R(f\ 0));$
$\bigwedge z.\ norm\ z \le R(f\ 0) \Longrightarrow f\ z \neq 0 \wedge f\ z \neq 1]\!] \Longrightarrow norm(deriv\ f\ 0) < 1$

theorem *little_Picard*:
assumes *holf*: f *holomorphic_on UNIV*
and $a \neq b$ *range* $f \cap \{a,b\} = \{\}$
obtains c **where** $f = (\lambda x.\ c)$

7.3 The Arzelà–Ascoli theorem

theorem *Arzela_Ascoli*:
fixes $\mathcal{F}$ :: $[nat,'a::euclidean\_space] \Rightarrow 'b::\{real\_normed\_vector,heine\_borel\}$
assumes *compact S*
and M: $\bigwedge n\ x.\ x \in S \Longrightarrow norm(\mathcal{F}\ n\ x) \le M$
and *equicont*:
$\bigwedge x\ e.\ [\![x \in S;\ 0 < e]\!]$
$\Longrightarrow \exists d.\ 0 < d \wedge (\forall n\ y.\ y \in S \wedge norm(x - y) < d \longrightarrow norm(\mathcal{F}\ n\ x - \mathcal{F}\ n\ y) < e)$
obtains $g\ k$ **where** *continuous_on S g strict_mono* $(k :: nat \Rightarrow nat)$
$\bigwedge e.\ 0 < e \Longrightarrow \exists N.\ \forall n\ x.\ n \ge N \wedge x \in S \longrightarrow norm(\mathcal{F}(k\ n)\ x - g\ x) < e$

7.3.1 Montel's theorem

theorem *Montel*:
fixes $\mathcal{F}$ :: $[nat,complex] \Rightarrow complex$
assumes *open S*
and $\mathcal{H}$: $\bigwedge h.\ h \in \mathcal{H} \Longrightarrow h$ *holomorphic_on S*
and *bounded*: $\bigwedge K.\ [\![compact\ K;\ K \subseteq S]\!] \Longrightarrow \exists B.\ \forall h \in \mathcal{H}.\ \forall\ z \in K.\ norm(h\ z) \le B$
and *rng_f*: *range* $\mathcal{F} \subseteq \mathcal{H}$
obtains $g\ r$
where g *holomorphic_on S strict_mono* $(r :: nat \Rightarrow nat)$
$\bigwedge x.\ x \in S \Longrightarrow ((\lambda n.\ \mathcal{F}\ (r\ n)\ x) \longrightarrow g\ x)$ *sequentially*
$\bigwedge K.\ [\![compact\ K;\ K \subseteq S]\!] \Longrightarrow uniform\_limit\ K\ (\mathcal{F} \circ r)\ g$ *sequentially*

7.4 Some simple but useful cases of Hurwitz's theorem

proposition *Hurwitz_no_zeros*:
assumes S: *open S connected S*

and *holf*: $\bigwedge n::nat.\ \mathcal{F}\ n$ *holomorphic_on S*
and *holg*: *g holomorphic_on S*
and *ul_g*: $\bigwedge K.\ [\![compact\ K;\ K \subseteq S]\!] \Longrightarrow$ *uniform_limit K* $\mathcal{F}$ *g sequentially*
and *nonconst*: $\neg$ *g constant_on S*
and *nz*: $\bigwedge n\ z.\ z \in S \Longrightarrow \mathcal{F}\ n\ z \neq 0$
and $z0 \in S$
shows $g\ z0 \neq 0$

corollary *Hurwitz_injective*:
assumes *S*: *open S connected S*
and *holf*: $\bigwedge n::nat.\ \mathcal{F}\ n$ *holomorphic_on S*
and *holg*: *g holomorphic_on S*
and *ul_g*: $\bigwedge K.\ [\![compact\ K;\ K \subseteq S]\!] \Longrightarrow$ *uniform_limit K* $\mathcal{F}$ *g sequentially*
and *nonconst*: $\neg$ *g constant_on S*
and *inj*: $\bigwedge n.$ *inj_on* $(\mathcal{F}\ n)$ *S*
shows *inj_on g S*

7.5 The Great Picard theorem

theorem *great_Picard*:
assumes *open M* $z \in M\ a \neq b$ **and** *holf*: *f holomorphic_on* $(M - \{z\})$
and *fab*: $\bigwedge w.\ w \in M - \{z\} \Longrightarrow f\ w \neq a \wedge f\ w \neq b$
obtains *l* **where** $(f \longrightarrow l)\ (at\ z) \vee ((inverse \circ f) \longrightarrow l)\ (at\ z)$

corollary *great_Picard_alt*:
assumes *M*: *open M* $z \in M$ **and** *holf*: *f holomorphic_on* $(M - \{z\})$
and *non*: $\bigwedge l.\ \neg\ (f \longrightarrow l)\ (at\ z)\ \bigwedge l.\ \neg\ ((inverse \circ f) \longrightarrow l)\ (at\ z)$
obtains *a* **where** $- \{a\} \subseteq f\ `\ (M - \{z\})$

corollary *great_Picard_infinite*:
assumes *M*: *open M* $z \in M$ **and** *holf*: *f holomorphic_on* $(M - \{z\})$
and *non*: $\bigwedge l.\ \neg\ (f \longrightarrow l)\ (at\ z)\ \bigwedge l.\ \neg\ ((inverse \circ f) \longrightarrow l)\ (at\ z)$
obtains *a* **where** $\bigwedge w.\ w \neq a \Longrightarrow$ *infinite* $\{x.\ x \in M - \{z\} \wedge f\ x = w\}$

theorem *Casorati_Weierstrass*:
assumes *open M* $z \in M$ *f holomorphic_on* $(M - \{z\})$
and $\bigwedge l.\ \neg\ (f \longrightarrow l)\ (at\ z)\ \bigwedge l.\ \neg\ ((inverse \circ f) \longrightarrow l)\ (at\ z)$
shows $closure(f\ `\ (M - \{z\})) = UNIV$

end

8 Moebius functions, Equivalents of Simply Connected Sets, Riemann Mapping Theorem

theory *Riemann_Mapping*
imports *Great_Picard*
begin

8.1 Moebius functions are biholomorphisms of the unit disc

definition *Moebius_function* :: [*real*,*complex*,*complex*] ⇒ *complex* **where**
Moebius_function ≡ λ*t w z*. *exp*(i ∗ *of_real t*) ∗ (*z* − *w*) / (*1* − *cnj w* ∗ *z*)

8.2 A big chain of equivalents of simple connectedness for an open set

proposition
assumes *open S*
shows *simply_connected_eq_winding_number_zero*:
simply_connected S ⟷
connected S ∧
(∀ *g z*. *path g* ∧ *path_image g* ⊆ *S* ∧
pathfinish g = *pathstart g* ∧ (*z* ∉ *S*)
⟶ *winding_number g z* = *0*) (**is** *?wn0*)
and *simply_connected_eq_contour_integral_zero*:
simply_connected S ⟷
connected S ∧
(∀ *g f*. *valid_path g* ∧ *path_image g* ⊆ *S* ∧
pathfinish g = *pathstart g* ∧ *f holomorphic_on S*
⟶ (*f has_contour_integral 0*) *g*) (**is** *?ci0*)
and *simply_connected_eq_global_primitive*:
simply_connected S ⟷
connected S ∧
(∀ *f*. *f holomorphic_on S* ⟶
(∃ *h*. ∀ *z*. *z* ∈ *S* ⟶ (*h has_field_derivative f z*) (*at z*))) (**is** *?gp*)
and *simply_connected_eq_holomorphic_log*:
simply_connected S ⟷
connected S ∧
(∀ *f*. *f holomorphic_on S* ∧ (∀ *z* ∈ *S*. *f z* ≠ *0*)
⟶ (∃ *g*. *g holomorphic_on S* ∧ (∀ *z* ∈ *S*. *f z* = *exp*(*g z*)))) (**is** *?log*)
and *simply_connected_eq_holomorphic_sqrt*:
simply_connected S ⟷
connected S ∧
(∀ *f*. *f holomorphic_on S* ∧ (∀ *z* ∈ *S*. *f z* ≠ *0*)
⟶ (∃ *g*. *g holomorphic_on S* ∧ (∀ *z* ∈ *S*. *f z* = (*g z*)^2))) (**is** *?sqrt*)
and *simply_connected_eq_biholomorphic_to_disc*:
simply_connected S ⟷

S = {} ∨ *S* = *UNIV* ∨
(∃ *f g*. *f holomorphic_on S* ∧ *g holomorphic_on ball 0 1* ∧
(∀ *z* ∈ *S*. *f z* ∈ *ball 0 1* ∧ *g*(*f z*) = *z*) ∧
(∀ *z* ∈ *ball 0 1*. *g z* ∈ *S* ∧ *f*(*g z*) = *z*)) (**is** *?bih*)
and *simply_connected_eq_homeomorphic_to_disc*:
simply_connected S ⟷ *S* = {} ∨ *S homeomorphic ball* (*0*::*complex*) *1*
(**is** *?disc*)

corollary *contractible_eq_simply_connected_2d*:
fixes *S* :: *complex set*
shows *open S* ⟹ (*contractible S* ⟷ *simply_connected S*)

8.3 A further chain of equivalences about components of the complement of a simply connected set

proposition
fixes *S* :: *complex set*
assumes *open S*
shows *simply_connected_eq_frontier_properties*:
simply_connected S ⟷
connected S ∧
(*if bounded S then connected*(*frontier S*)
else (∀ *C* ∈ *components*(*frontier S*). ¬*bounded C*)) (**is** *?fp*)
and *simply_connected_eq_unbounded_complement_components*:
simply_connected S ⟷
connected S ∧ (∀ *C* ∈ *components*(− *S*). ¬*bounded C*) (**is** *?ucc*)
and *simply_connected_eq_empty_inside*:
simply_connected S ⟷
connected S ∧ *inside S* = {} (**is** *?ei*)

8.4 Further equivalences based on continuous logs and sqrts

proposition
fixes *S* :: *complex set*
assumes *open S*
shows *simply_connected_eq_continuous_log*:
simply_connected S ⟷
connected S ∧
(∀ *f*::*complex*⇒*complex*. *continuous_on S f* ∧ (∀ *z* ∈ *S*. *f z* ≠ *0*)
⟶ (∃ *g*. *continuous_on S g* ∧ (∀ *z* ∈ *S*. *f z* = *exp* (*g z*)))) (**is** *?log*)
and *simply_connected_eq_continuous_sqrt*:
simply_connected S ⟷
connected S ∧
(∀ *f*::*complex*⇒*complex*. *continuous_on S f* ∧ (∀ *z* ∈ *S*. *f z* ≠ *0*)

$\longrightarrow (\exists\, g.\ continuous\_on\ S\ g \wedge (\forall\, z \in S.\ f\ z = (g\ z)^2)))$ (**is** *?sqrt*)

8.5 Finally, the Riemann Mapping Theorem

theorem *Riemann_mapping_theorem*:
$open\ S \wedge simply\_connected\ S \longleftrightarrow$
$S = \{\} \vee S = UNIV \vee$
$(\exists\, f\ g.\ f\ holomorphic\_on\ S \wedge g\ holomorphic\_on\ ball\ 0\ 1 \wedge$
$(\forall\, z \in S.\ f\ z \in ball\ 0\ 1 \wedge g(f\ z) = z) \wedge$
$(\forall\, z \in ball\ 0\ 1.\ g\ z \in S \wedge f(g\ z) = z))$
(**is** _ = *?rhs*)

8.6 Applications to Winding Numbers

8.7 Winding number equality is the same as path/loop homotopy in C - 0

proposition *winding_number_homotopic_paths_eq*:
assumes *path p* **and** ζp: $\zeta \notin path\_image\ p$
and *path q* **and** ζq: $\zeta \notin path\_image\ q$
and *qp*: $pathstart\ q = pathstart\ p\ pathfinish\ q = pathfinish\ p$
shows $winding\_number\ p\ \zeta = winding\_number\ q\ \zeta \longleftrightarrow homotopic\_paths\ (-\{\zeta\})\ p\ q$
(**is** *?lhs* = *?rhs*)

end
theory *Meromorphic*
imports *Laurent_Convergence Riemann_Mapping*
begin

theorem *argument_principle′*:
fixes f::*complex* ⇒ *complex* **and** *poles s*:: *complex set*
— *pz* is the set of non-essential singularities and zeros
defines $pz \equiv \{w \in s.\ f\ w = 0 \vee w \in poles\}$
assumes *open s* **and**
connected s **and**
f_holo:f *holomorphic_on* $s - poles$ **and**
h_holo:*h holomorphic_on s* **and**
valid_path g **and**
loop:$pathfinish\ g = pathstart\ g$ **and**
path_img:$path\_image\ g \subseteq s - pz$ **and**
homo:$\forall\, z.\ (z \notin s) \longrightarrow winding\_number\ g\ z = 0$ **and**

finite:*finite pz* **and**
poles:$\forall$ *p*$\in$*s*$\cap$*poles. not_essential f p*
shows *contour_integral g* (λ*x. deriv f x* $*$ *h x* / *f x*) = *2* $*$ *pi* $*$ i $*$
($\sum$ *p*$\in$*pz. winding_number g p* $*$ *h p* $*$ *zorder f p*)

theorem *Residue_theorem_inside*:
assumes *f*: *f meromorphic_on s pts*
simply_connected s
assumes *g*: *valid_path g*
pathfinish g = *pathstart g*
path_image g $\subseteq$ *s* − *pts*
defines *pts1* $\equiv$ *pts* $\cap$ *inside* (*path_image g*)
shows *finite pts1*
and *contour_integral g f* = *2* $*$ *pi* $*$ i $*$ ($\sum$ *p*$\in$*pts1*. *winding_number g p* $*$
residue f p)

theorem *Residue_theorem'*:
assumes *f*: *f meromorphic_on s pts*
simply_connected s
assumes *g*: *valid_path g*
pathfinish g = *pathstart g*
path_image g $\subseteq$ *s* − *pts*
assumes *pts'*: *finite pts'*
pts' $\subseteq$ *s*
$\bigwedge$*z*. *z* $\in$ *pts* − *pts'* $\Longrightarrow$ *winding_number g z* = *0*
shows *contour_integral g f* = *2* $*$ *pi* $*$ i $*$ ($\sum$ *p*$\in$*pts'*. *winding_number g p* $*$
residue f p)

end
theory *Complex_Analysis*
imports
Residue_Theorem
Meromorphic
begin

end

References

[1]