Functional Data Structures

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Abstract

A collection of verified functional data structures. The emphasis is on conciseness of algorithms and succinctness of proofs, more in the style of a textbook than a library of efficient algorithms.

For more details see [13].

Contents

1 Sorting

theory *Sorting*
imports
  Complex_Main
  HOL−Library.Multiset
begin

hide_const *List.insort*

declare *Let_def* [*simp*]

1.1 Insertion Sort

fun *insort1* :: *'a::linorder* ⇒ *'a list* ⇒ *'a list* **where**
insort1 x [] = [*x*] |
insort1 x (*y*#*ys*) =
  (*if x* ≤ *y then x*#*y*#*ys else y*#(*insort1 x ys*))

fun *insort* :: *'a::linorder list* ⇒ *'a list* **where**
insort [] = [] |
insort (*x*#*xs*) = *insort1 x* (*insort xs*)

1.1.1 Functional Correctness

lemma *mset_insort1*: *mset* (*insort1 x xs*) = {#*x*#} + *mset xs*
apply(*induction xs*)
apply *auto*
done

lemma *mset_insort*: *mset* (*insort xs*) = *mset xs*
apply(*induction xs*)
apply *simp*
apply (*simp add*: *mset_insort1*)
done

lemma *set_insort1*: *set* (*insort1 x xs*) = {*x*} ∪ *set xs*
by(*simp add*: *mset_insort1 flip*: *set_mset_mset*)

lemma *sorted_insort1*: *sorted* (*insort1 a xs*) = *sorted xs*
apply(*induction xs*)
apply(*auto simp add*: *set_insort1*)
done

lemma *sorted_insort*: *sorted* (*insort xs*)

apply(*induction xs*)
apply(*auto simp*: *sorted_insort1*)
done

1.1.2 Time Complexity

We count the number of function calls.

insort1 x [] = [*x*] *insort1 x* (*y*#*ys*) = (*if x* ≤ *y then x*#*y*#*ys else y*#(*insort1 x ys*))

fun *T_insort1* :: *'a*::*linorder* ⇒ *'a list* ⇒ *nat* **where**
T_insort1 x [] = *1* |
T_insort1 x (*y*#*ys*) =
 (*if x* ≤ *y then 0 else T_insort1 x ys*) + *1*

insort [] = [] *insort* (*x*#*xs*) = *insort1 x* (*insort xs*)

fun *T_insort* :: *'a*::*linorder list* ⇒ *nat* **where**
T_insort [] = *1* |
T_insort (*x*#*xs*) = *T_insort xs* + *T_insort1 x* (*insort xs*) + *1*

lemma *T_insort1_length*: *T_insort1 x xs* ≤ *length xs* + *1*
apply(*induction xs*)
apply *auto*
done

lemma *length_insort1*: *length* (*insort1 x xs*) = *length xs* + *1*
apply(*induction xs*)
apply *auto*
done

lemma *length_insort*: *length* (*insort xs*) = *length xs*
apply(*induction xs*)
apply (*auto simp*: *length_insort1*)
done

lemma *T_insort_length*: *T_insort xs* ≤ (*length xs* + *1*) ^ *2*
proof(*induction xs*)
 case *Nil* **show** *?case* **by** *simp*
next
 case (*Cons x xs*)
 have *T_insort* (*x*#*xs*) = *T_insort xs* + *T_insort1 x* (*insort xs*) + *1* **by** *simp*
 also have ... ≤ (*length xs* + *1*) ^ *2* + *T_insort1 x* (*insort xs*) + *1*

```
    using Cons.IH by simp
  also have ... ≤ (length xs + 1) ^ 2 + length xs + 1 + 1
    using T_insort1_length[of x insort xs] by (simp add: length_insort)
  also have ... ≤ (length(x#xs) + 1) ^ 2
    by (simp add: power2_eq_square)
  finally show ?case .
qed
```

1.2 Merge Sort

```
fun merge :: 'a::linorder list ⇒ 'a list ⇒ 'a list where
merge [] ys = ys |
merge xs [] = xs |
merge (x#xs) (y#ys) = (if x ≤ y then x # merge xs (y#ys) else y # merge
(x#xs) ys)

fun msort :: 'a::linorder list ⇒ 'a list where
msort xs = (let n = length xs in
  if n ≤ 1 then xs
  else merge (msort (take (n div 2) xs)) (msort (drop (n div 2) xs)))

declare msort.simps [simp del]
```

1.2.1 Functional Correctness

```
lemma mset_merge: mset(merge xs ys) = mset xs + mset ys
by(induction xs ys rule: merge.induct) auto

lemma mset_msort: mset (msort xs) = mset xs
proof(induction xs rule: msort.induct)
  case (1 xs)
  let ?n = length xs
  let ?ys = take (?n div 2) xs
  let ?zs = drop (?n div 2) xs
  show ?case
  proof cases
    assume ?n ≤ 1
    thus ?thesis by(simp add: msort.simps[of xs])
  next
    assume ¬ ?n ≤ 1
    hence mset (msort xs) = mset (msort ?ys) + mset (msort ?zs)
      by(simp add: msort.simps[of xs] mset_merge)
    also have ... = mset ?ys + mset ?zs
      using ‹¬ ?n ≤ 1› by(simp add: 1.IH)
```

```
  also have ... = mset (?ys @ ?zs) by (simp del: append_take_drop_id)
  also have ... = mset xs by simp
  finally show ?thesis .
 qed
qed
```

Via the previous lemma or directly:

```
lemma set_merge: set(merge xs ys) = set xs ∪ set ys
by (metis mset_merge set_mset_mset set_mset_union)

lemma set(merge xs ys) = set xs ∪ set ys
by(induction xs ys rule: merge.induct) (auto)

lemma sorted_merge: sorted (merge xs ys) ⟷ (sorted xs ∧ sorted ys)
by(induction xs ys rule: merge.induct) (auto simp: set_merge)

lemma sorted_msort: sorted (msort xs)
proof(induction xs rule: msort.induct)
 case (1 xs)
 let ?n = length xs
 show ?case
 proof cases
  assume ?n ≤ 1
  thus ?thesis by(simp add: msort.simps[of xs] sorted01)
 next
  assume ¬ ?n ≤ 1
  thus ?thesis using 1.IH
   by(simp add: sorted_merge msort.simps[of xs])
 qed
qed
```

1.2.2 Time Complexity

We only count the number of comparisons between list elements.

```
fun C_merge :: 'a::linorder list ⇒ 'a list ⇒ nat where
C_merge [] ys = 0 |
C_merge xs [] = 0 |
C_merge (x#xs) (y#ys) = 1 + (if x ≤ y then C_merge xs (y#ys) else
C_merge (x#xs) ys)

lemma C_merge_ub: C_merge xs ys ≤ length xs + length ys
by (induction xs ys rule: C_merge.induct) auto

fun C_msort :: 'a::linorder list ⇒ nat where
```

C_msort xs =
(let n = length xs;
ys = take (n div 2) xs;
zs = drop (n div 2) xs
in if n ≤ 1 then 0
else C_msort ys + C_msort zs + C_merge (msort ys) (msort zs))

declare *C_msort.simps [simp del]*

lemma *length_merge: length(merge xs ys) = length xs + length ys*
apply *(induction xs ys rule: merge.induct)*
apply *auto*
done

lemma *length_msort: length(msort xs) = length xs*
proof *(induction xs rule: msort.induct)*
case *(1 xs)*
show *?case*
by *(auto simp: msort.simps [of xs] 1 length_merge)*
qed

Why structured proof? To have the name "xs" to specialize msort.simps with xs to ensure that msort.simps cannot be used recursively. Also works without this precaution, but that is just luck.

lemma *C_msort_le: length xs = 2^k ⟹ C_msort xs ≤ k * 2^k*
proof*(induction k arbitrary: xs)*
case *0* **thus** *?case* **by** *(simp add: C_msort.simps)*
next
case *(Suc k)*
let *?n = length xs*
let *?ys = take (?n div 2) xs*
let *?zs = drop (?n div 2) xs*
show *?case*
proof *(cases ?n ≤ 1)*
case *True*
thus *?thesis* **by***(simp add: C_msort.simps)*
next
case *False*
have *C_msort(xs) =*
C_msort ?ys + C_msort ?zs + C_merge (msort ?ys) (msort ?zs)
by *(simp add: C_msort.simps msort.simps)*
also have *... ≤ C_msort ?ys + C_msort ?zs + length ?ys + length ?zs*
using *C_merge_ub[of msort ?ys msort ?zs] length_msort[of ?ys]*

length_msort[*of ?zs*]
  by *arith*
 also have ... $\leq$ *k ∗ 2^k + C_msort ?zs + length ?ys + length ?zs*
  using *Suc.IH*[*of ?ys*] *Suc.prems* **by** *simp*
 also have ... $\leq$ *k ∗ 2^k + k ∗ 2^k + length ?ys + length ?zs*
  using *Suc.IH*[*of ?zs*] *Suc.prems* **by** *simp*
 also have ... = *2 ∗ k ∗ 2^k + 2 ∗ 2 ^ k*
  using *Suc.prems* **by** *simp*
 finally show *?thesis* **by** *simp*
 qed
qed

lemma *C_msort_log*: *length xs = 2^k* $\Longrightarrow$ *C_msort xs* $\leq$ *length xs ∗ log 2* (*length xs*)
using *C_msort_le*[*of xs k*] **apply** (*simp add*: *log_nat_power algebra_simps*)
by (*metis* (*mono_tags*) *numeral_power_eq_of_nat_cancel_iff of_nat_le_iff of_nat_mult*)

1.3 Bottom-Up Merge Sort

fun *merge_adj* :: (*'a*::*linorder*) *list list* ⇒ *'a list list* **where**
merge_adj [] = [] |
merge_adj [*xs*] = [*xs*] |
merge_adj (*xs* # *ys* # *zss*) = *merge xs ys* # *merge_adj zss*

For the termination proof of *merge_all* below.

lemma *length_merge_adjacent*[*simp*]: *length* (*merge_adj xs*) = (*length xs* + *1*) *div 2*
by (*induction xs rule*: *merge_adj.induct*) *auto*

fun *merge_all* :: (*'a*::*linorder*) *list list* ⇒ *'a list* **where**
merge_all [] = [] |
merge_all [*xs*] = *xs* |
merge_all xss = *merge_all* (*merge_adj xss*)

definition *msort_bu* :: (*'a*::*linorder*) *list* ⇒ *'a list* **where**
msort_bu xs = *merge_all* (*map* (*λx.* [*x*]) *xs*)

1.3.1 Functional Correctness

abbreviation *mset_mset* :: *'a list list* ⇒ *'a multiset* **where**
mset_mset xss ≡ $\sum_{\#}$ (*image_mset mset* (*mset xss*))

lemma *mset_merge_adj*:

$mset\_mset\ (merge\_adj\ xss) = mset\_mset\ xss$
by(*induction xss rule*: *merge_adj.induct*) (*auto simp*: *mset_merge*)

lemma *mset_merge_all*:
$mset\ (merge\_all\ xss) = mset\_mset\ xss$
by(*induction xss rule*: *merge_all.induct*) (*auto simp*: *mset_merge mset_merge_adj*)

lemma *mset_msort_bu*: $mset\ (msort\_bu\ xs) = mset\ xs$
by(*simp add*: *msort_bu_def mset_merge_all multiset.map_comp comp_def*)

lemma *sorted_merge_adj*:
$\forall xs \in set\ xss.\ sorted\ xs \Longrightarrow \forall xs \in set\ (merge\_adj\ xss).\ sorted\ xs$
by(*induction xss rule*: *merge_adj.induct*) (*auto simp*: *sorted_merge*)

lemma *sorted_merge_all*:
$\forall xs \in set\ xss.\ sorted\ xs \Longrightarrow sorted\ (merge\_all\ xss)$
apply(*induction xss rule*: *merge_all.induct*)
using [[*simp_depth_limit=3*]] **by** (*auto simp add*: *sorted_merge_adj*)

lemma *sorted_msort_bu*: $sorted\ (msort\_bu\ xs)$
by(*simp add*: *msort_bu_def sorted_merge_all*)

1.3.2 Time Complexity

fun *C_merge_adj* :: ('*a*::*linorder*) *list list* ⇒ *nat* **where**
$C\_merge\_adj\ [] = 0\ |$
$C\_merge\_adj\ [xs] = 0\ |$
$C\_merge\_adj\ (xs\ \#\ ys\ \#\ zss) = C\_merge\ xs\ ys + C\_merge\_adj\ zss$

fun *C_merge_all* :: ('*a*::*linorder*) *list list* ⇒ *nat* **where**
$C\_merge\_all\ [] = 0\ |$
$C\_merge\_all\ [xs] = 0\ |$
$C\_merge\_all\ xss = C\_merge\_adj\ xss + C\_merge\_all\ (merge\_adj\ xss)$

definition *C_msort_bu* :: ('*a*::*linorder*) *list* ⇒ *nat* **where**
$C\_msort\_bu\ xs = C\_merge\_all\ (map\ (\lambda x.\ [x])\ xs)$

lemma *length_merge_adj*:
$[\![\ even(length\ xss);\ \forall xs \in set\ xss.\ length\ xs = m\ ]\!]$
$\Longrightarrow \forall xs \in set\ (merge\_adj\ xss).\ length\ xs = 2*m$
by(*induction xss rule*: *merge_adj.induct*) (*auto simp*: *length_merge*)

lemma *C_merge_adj*: $\forall xs \in set\ xss.\ length\ xs = m \Longrightarrow C\_merge\_adj\ xss$
$\le m * length\ xss$

proof(*induction xss rule*: *C_merge_adj.induct*)
  case *1* **thus** *?case* **by** *simp*
next
  case *2* **thus** *?case* **by** *simp*
next
  case (*3 x y*) **thus** *?case* **using** *C_merge_ub*[*of x y*] **by** (*simp add*: *algebra_simps*)
qed

lemma *C_merge_all*: ⟦ ∀ *xs* ∈ *set xss*. *length xs* = *m*; *length xss* = *2^k* ⟧
  ⟹ *C_merge_all xss* ≤ *m* ∗ *k* ∗ *2^k*
proof (*induction xss arbitrary*: *k m rule*: *C_merge_all.induct*)
  case *1* **thus** *?case* **by** *simp*
next
  case *2* **thus** *?case* **by** *simp*
next
  case (*3 xs ys xss*)
  let *?xss* = *xs* # *ys* # *xss*
  let *?xss2* = *merge_adj ?xss*
  obtain *k′* **where** *k′*: *k* = *Suc k′* **using** *3.prems*(*2*)
   by (*metis length_Cons nat.inject nat_power_eq_Suc_0_iff nat.exhaust*)
  have *even* (*length ?xss*) **using** *3.prems*(*2*) *k′* **by** *auto*
  from *length_merge_adj*[*OF this 3.prems*(*1*)]
  have ∗: ∀ *x* ∈ *set*(*merge_adj ?xss*). *length x* = *2*∗*m* **.**
  have ∗∗: *length ?xss2* = *2* ^ *k′* **using** *3.prems*(*2*) *k′* **by** *auto*
  have *C_merge_all ?xss* = *C_merge_adj ?xss* + *C_merge_all ?xss2* **by** *simp*
  also have . . . ≤ *m* ∗ *2^k* + *C_merge_all ?xss2*
    using *3.prems*(*2*) *C_merge_adj*[*OF 3.prems*(*1*)] **by** (*auto simp*: *algebra_simps*)
  also have . . . ≤ *m* ∗ *2^k* + (*2*∗*m*) ∗ *k′* ∗ *2^k′*
    using *3.IH*[*OF* ∗ ∗∗] **by** *simp*
  also have . . . = *m* ∗ *k* ∗ *2^k*
    using *k′* **by** (*simp add*: *algebra_simps*)
  finally show *?case* **.**
qed

corollary *C_msort_bu*: *length xs* = *2* ^ *k* ⟹ *C_msort_bu xs* ≤ *k* ∗ *2* ^ *k*
using *C_merge_all*[*of map* (λ*x*. [*x*]) *xs 1*] **by** (*simp add*: *C_msort_bu_def*)

1.4 Quicksort

fun *quicksort* :: (*′a*::*linorder*) *list* ⇒ *′a list* **where**

quicksort [] = [] |
quicksort $(x\#xs)$ = *quicksort* (*filter* $(\lambda y.\ y < x)$ *xs*) @ [*x*] @ *quicksort* (*filter* $(\lambda y.\ x \le y)$ *xs*)

lemma *mset_quicksort*: *mset* (*quicksort xs*) = *mset xs*
apply (*induction xs rule*: *quicksort.induct*)
apply (*auto simp*: *not_le*)
done

lemma *set_quicksort*: *set* (*quicksort xs*) = *set xs*
by(*rule mset_eq_setD*[*OF mset_quicksort*])

lemma *sorted_quicksort*: *sorted* (*quicksort xs*)
apply (*induction xs rule*: *quicksort.induct*)
apply (*auto simp add*: *sorted_append set_quicksort*)
done

1.5 Insertion Sort w.r.t. Keys and Stability

hide_const *List.insort_key*

fun *insort1_key* :: $('a \Rightarrow 'k{::}linorder) \Rightarrow 'a \Rightarrow 'a\ list \Rightarrow 'a\ list$ **where**
insort1_key f x [] = [*x*] |
insort1_key f x (*y* # *ys*) = (*if f x* ≤ *f y then x* # *y* # *ys else y* # *insort1_key f x ys*)

fun *insort_key* :: $('a \Rightarrow 'k{::}linorder) \Rightarrow 'a\ list \Rightarrow 'a\ list$ **where**
insort_key f [] = [] |
insort_key f (*x* # *xs*) = *insort1_key f x* (*insort_key f xs*)

1.5.1 Standard functional correctness

lemma *mset_insort1_key*: *mset* (*insort1_key f x xs*) = {#*x*#} + *mset xs*
by(*induction xs*) *simp_all*

lemma *mset_insort_key*: *mset* (*insort_key f xs*) = *mset xs*
by(*induction xs*) (*simp_all add*: *mset_insort1_key*)

lemma *set_insort1_key*: *set* (*insort1_key f x xs*) = {*x*} ∪ *set xs*
by (*induction xs*) *auto*

lemma *sorted_insort1_key*: *sorted* (*map f* (*insort1_key f a xs*)) = *sorted* (*map f xs*)

by(*induction xs*)(*auto simp*: *set_insort1_key*)

lemma *sorted_insort_key*: *sorted* (*map f* (*insort_key f xs*))
by(*induction xs*)(*simp_all add*: *sorted_insort1_key*)

1.5.2 Stability

lemma *insort1_is_Cons*: $\forall$ *x*$\in$*set xs*. *f a* $\leq$ *f x* $\Longrightarrow$ *insort1_key f a xs* = *a* # *xs*
by (*cases xs*) *auto*

lemma *filter_insort1_key_neg*:
$\neg$ *P x* $\Longrightarrow$ *filter P* (*insort1_key f x xs*) = *filter P xs*
by (*induction xs*) *simp_all*

lemma *filter_insort1_key_pos*:
sorted (*map f xs*) $\Longrightarrow$ *P x* $\Longrightarrow$ *filter P* (*insort1_key f x xs*) = *insort1_key f x* (*filter P xs*)
by (*induction xs*) (*auto*, *subst insort1_is_Cons*, *auto*)

lemma *sort_key_stable*: *filter* (λ*y*. *f y* = *k*) (*insort_key f xs*) = *filter* (λ*y*. *f y* = *k*) *xs*
proof (*induction xs*)
  case *Nil* **thus** *?case* **by** *simp*
next
  case (*Cons a xs*)
  thus *?case*
  proof (*cases f a* = *k*)
   case *False* **thus** *?thesis* **by** (*simp add*: *Cons.IH filter_insort1_key_neg*)
  next
    case *True*
    have *filter* (λ*y*. *f y* = *k*) (*insort_key f* (*a* # *xs*))
      = *filter* (λ*y*. *f y* = *k*) (*insort1_key f a* (*insort_key f xs*)) **by** *simp*
    also have . . . = *insort1_key f a* (*filter* (λ*y*. *f y* = *k*) (*insort_key f xs*))
      by (*simp add*: *True filter_insort1_key_pos sorted_insort_key*)
    also have . . . = *insort1_key f a* (*filter* (λ*y*. *f y* = *k*) *xs*) **by** (*simp add*: *Cons.IH*)
    also have . . . = *a* # (*filter* (λ*y*. *f y* = *k*) *xs*) **by**(*simp add*: *True insort1_is_Cons*)
    also have . . . = *filter* (λ*y*. *f y* = *k*) (*a* # *xs*) **by** (*simp add*: *True*)
    finally show *?thesis* **.**
  qed
qed

1.6 Uniqueness of Sorting

```
lemma sorting_unique:
  assumes mset ys = mset xs sorted xs sorted ys
  shows xs = ys
  using assms
proof (induction xs arbitrary: ys)
  case (Cons x xs ys')
  obtain y ys where ys': ys' = y # ys
    using Cons.prems by (cases ys') auto
  have x = y
    using Cons.prems unfolding ys'
  proof (induction x y arbitrary: xs ys rule: linorder_wlog)
    case (le x y xs ys)
    have x ∈# mset (x # xs)
      by simp
    also have mset (x # xs) = mset (y # ys)
      using le by simp
    finally show x = y
      using le by auto
  qed (simp_all add: eq_commute)
  thus ?case
    using Cons.prems Cons.IH[of ys] by (auto simp: ys')
qed auto


end
```

2 Creating Almost Complete Trees

```
theory Balance
imports
  HOL−Library.Tree_Real
begin

fun bal :: nat ⇒ 'a list ⇒ 'a tree ∗ 'a list where
bal n xs = (if n=0 then (Leaf,xs) else
 (let m = n div 2;
      (l, ys) = bal m xs;
      (r, zs) = bal (n−1−m) (tl ys)
  in (Node l (hd ys) r, zs)))

declare bal.simps[simp del]
declare Let_def[simp]
```

definition *bal_list* :: *nat* ⇒ *'a list* ⇒ *'a tree* **where**
bal_list n xs = *fst* (*bal n xs*)

definition *balance_list* :: *'a list* ⇒ *'a tree* **where**
balance_list xs = *bal_list* (*length xs*) *xs*

definition *bal_tree* :: *nat* ⇒ *'a tree* ⇒ *'a tree* **where**
bal_tree n t = *bal_list n* (*inorder t*)

definition *balance_tree* :: *'a tree* ⇒ *'a tree* **where**
balance_tree t = *bal_tree* (*size t*) *t*

lemma *bal_simps*:
  bal 0 xs = (*Leaf*, *xs*)
  n > *0* ⟹
   bal n xs =
  (*let m* = *n div 2*;
     (*l*, *ys*) = *bal m xs*;
     (*r*, *zs*) = *bal* (*n*−*1*−*m*) (*tl ys*)
  in (*Node l* (*hd ys*) *r*, *zs*))
by(*simp_all add*: *bal.simps*)

lemma *bal_inorder*:
  ⟦ *n* ≤ *length xs*; *bal n xs* = (*t*,*zs*) ⟧
  ⟹ *xs* = *inorder t* @ *zs* ∧ *size t* = *n*
proof(*induction n arbitrary*: *xs t zs rule*: *less_induct*)
  case (*less n*) **show** *?case*
  proof *cases*
   assume *n* = *0* **thus** *?thesis* **using** *less.prems* **by** (*simp add*: *bal_simps*)
  next
    assume [*arith*]: *n* ≠ *0*
    let *?m* = *n div 2* **let** *?m'* = *n* − *1* − *?m*
    from *less.prems*(*2*) **obtain** *l r ys* **where**
      b1: *bal ?m xs* = (*l*,*ys*) **and**
      b2: *bal ?m'* (*tl ys*) = (*r*,*zs*) **and**
      t: *t* = ⟨*l*, *hd ys*, *r*⟩
      by(*auto simp*: *bal_simps split*: *prod.splits*)
    have *IH1*: *xs* = *inorder l* @ *ys* ∧ *size l* = *?m*
      using *b1 less.prems*(*1*) **by**(*intro less.IH*) *auto*
    have *IH2*: *tl ys* = *inorder r* @ *zs* ∧ *size r* = *?m'*
      using *b2 IH1 less.prems*(*1*) **by**(*intro less.IH*) *auto*
     show *?thesis* **using** *t IH1 IH2 less.prems*(*1*) *hd_Cons_tl*[*of ys*] **by**
fastforce

qed
qed

corollary *inorder_bal_list*[*simp*]:
$n \le$ *length xs* $\Longrightarrow$ *inorder*(*bal_list n xs*) = *take n xs*
unfolding *bal_list_def*
by (*metis* (*mono_tags*) *prod.collapse*[*of bal n xs*] *append_eq_conv_conj bal_inorder length_inorder*)

corollary *inorder_balance_list*[*simp*]: *inorder*(*balance_list xs*) = *xs*
by(*simp add*: *balance_list_def*)

corollary *inorder_bal_tree*:
$n \le$ *size t* $\Longrightarrow$ *inorder*(*bal_tree n t*) = *take n* (*inorder t*)
by(*simp add*: *bal_tree_def*)

corollary *inorder_balance_tree*[*simp*]: *inorder*(*balance_tree t*) = *inorder t*
by(*simp add*: *balance_tree_def inorder_bal_tree*)

The length/size lemmas below do not require the precondition $n \le$ *length xs* (or $n \le$ *size t*) that they come with. They could take advantage of the fact that *bal xs n* yields a result even if *length xs* $< n$. In that case the result will contain one or more occurrences of *hd* []. However, this is counter-intuitive and does not reflect the execution in an eager functional language.

lemma *bal_length*: ⟦ $n \le$ *length xs*; *bal n xs* = (*t*,*zs*) ⟧ $\Longrightarrow$ *length zs* = *length xs* − *n*
using *bal_inorder* **by** *fastforce*

corollary *size_bal_list*[*simp*]: $n \le$ *length xs* $\Longrightarrow$ *size*(*bal_list n xs*) = *n*
unfolding *bal_list_def* **using** *bal_inorder prod.exhaust_sel* **by** *blast*

corollary *size_balance_list*[*simp*]: *size*(*balance_list xs*) = *length xs*
by (*simp add*: *balance_list_def*)

corollary *size_bal_tree*[*simp*]: $n \le$ *size t* $\Longrightarrow$ *size*(*bal_tree n t*) = *n*
by(*simp add*: *bal_tree_def*)

corollary *size_balance_tree*[*simp*]: *size*(*balance_tree t*) = *size t*
by(*simp add*: *balance_tree_def*)

lemma *min_height_bal*:
⟦ $n \le$ *length xs*; *bal n xs* = (*t*,*zs*) ⟧ $\Longrightarrow$ *min_height t* = *nat*(⌊*log 2* (*n* + *1*)⌋)
proof(*induction n arbitrary*: *xs t zs rule*: *less_induct*)

```
  case (less n)
  show ?case
  proof cases
   assume n = 0 thus ?thesis using less.prems(2) by (simp add: bal_simps)
  next
    assume [arith]: n ≠ 0
    let ?m = n div 2 let ?m' = n − 1 − ?m
    from less.prems obtain l r ys where
      b1: bal ?m xs = (l,ys) and
      b2: bal ?m' (tl ys) = (r,zs) and
      t: t = ⟨l, hd ys, r⟩
      by(auto simp: bal_simps split: prod.splits)
    let ?hl = nat (floor(log 2 (?m + 1)))
    let ?hr = nat (floor(log 2 (?m' + 1)))
    have IH1: min_height l = ?hl using less.IH[OF _ _ b1] less.prems(1)
by simp
    have IH2: min_height r = ?hr
      using less.prems(1) bal_length[OF _ b1] b2 by(intro less.IH) auto
    have (n+1) div 2 ≥ 1 by arith
    hence 0: log 2 ((n+1) div 2) ≥ 0 by simp
    have ?m' ≤ ?m by arith
    hence le: ?hr ≤ ?hl by(simp add: nat_mono floor_mono)
    have min_height t = min ?hl ?hr + 1 by (simp add: t IH1 IH2)
    also have ... = ?hr + 1 using le by (simp add: min_absorb2)
    also have ?m' + 1 = (n+1) div 2 by linarith
    also have nat (floor(log 2 ((n+1) div 2))) + 1
       = nat (floor(log 2 ((n+1) div 2) + 1))
      using 0 by linarith
    also have ... = nat (floor(log 2 (n + 1)))
      using floor_log2_div2[of n+1] by (simp add: log_mult)
    finally show ?thesis .
  qed
qed

lemma height_bal:
  ⟦ n ≤ length xs; bal n xs = (t,zs) ⟧ ⟹ height t = nat ⌈log 2 (n + 1)⌉
proof(induction n arbitrary: xs t zs rule: less_induct)
  case (less n) show ?case
  proof cases
    assume n = 0 thus ?thesis
      using less.prems by (simp add: bal_simps)
  next
    assume [arith]: n ≠ 0
    let ?m = n div 2 let ?m' = n − 1 − ?m
```

```
    from less.prems obtain l r ys where
      b1: bal ?m xs = (l,ys) and
      b2: bal ?m' (tl ys) = (r,zs) and
      t: t = ⟨l, hd ys, r⟩
      by(auto simp: bal_simps split: prod.splits)
    let ?hl = nat ⌈log 2 (?m + 1)⌉
    let ?hr = nat ⌈log 2 (?m' + 1)⌉
    have IH1: height l = ?hl using less.IH[OF _ _ b1] less.prems(1) by
simp
    have IH2: height r = ?hr
      using b2 bal_length[OF _ b1] less.prems(1) by(intro less.IH) auto
    have 0: log 2 (?m + 1) ≥ 0 by simp
    have ?m' ≤ ?m by arith
    hence le: ?hr ≤ ?hl
      by(simp add: nat_mono ceiling_mono del: nat_ceiling_le_eq)
    have height t = max ?hl ?hr + 1 by (simp add: t IH1 IH2)
    also have ... = ?hl + 1 using le by (simp add: max_absorb1)
    also have ... = nat ⌈log 2 (?m + 1) + 1⌉ using 0 by linarith
    also have ... = nat ⌈log 2 (n + 1)⌉
      using ceiling_log2_div2[of n+1] by (simp)
    finally show ?thesis .
  qed
qed

lemma acomplete_bal:
  assumes n ≤ length xs bal n xs = (t,ys) shows acomplete t
unfolding acomplete_def
using height_bal[OF assms] min_height_bal[OF assms]
by linarith

lemma height_bal_list:
  n ≤ length xs ⟹ height (bal_list n xs) = nat ⌈log 2 (n + 1)⌉
unfolding bal_list_def by (metis height_bal prod.collapse)

lemma height_balance_list:
  height (balance_list xs) = nat ⌈log 2 (length xs + 1)⌉
by (simp add: balance_list_def height_bal_list)

corollary height_bal_tree:
  n ≤ size t ⟹ height (bal_tree n t) = nat⌈log 2 (n + 1)⌉
unfolding bal_list_def bal_tree_def
by (metis bal_list_def height_bal_list length_inorder)

corollary height_balance_tree:
```

height (*balance_tree t*) = *nat*⌈*log 2* (*size t* + *1*)⌉
by (*simp add*: *bal_tree_def balance_tree_def height_bal_list*)

corollary *acomplete_bal_list*[*simp*]: *n* ≤ *length xs* ⟹ *acomplete* (*bal_list n xs*)
unfolding *bal_list_def* **by** (*metis acomplete_bal prod.collapse*)

corollary *acomplete_balance_list*[*simp*]: *acomplete* (*balance_list xs*)
by (*simp add*: *balance_list_def*)

corollary *acomplete_bal_tree*[*simp*]: *n* ≤ *size t* ⟹ *acomplete* (*bal_tree n t*)
by (*simp add*: *bal_tree_def*)

corollary *acomplete_balance_tree*[*simp*]: *acomplete* (*balance_tree t*)
by (*simp add*: *balance_tree_def*)

lemma *wbalanced_bal*: ⟦ *n* ≤ *length xs*; *bal n xs* = (*t*,*ys*) ⟧ ⟹ *wbalanced t*
proof(*induction n arbitrary*: *xs t ys rule*: *less_induct*)
  case (*less n*)
  show *?case*
  proof *cases*
    assume *n* = *0*
    thus *?thesis* **using** *less.prems*(*2*) **by**(*simp add*: *bal_simps*)
  next
    assume [*arith*]: *n* ≠ *0*
    with *less.prems* **obtain** *l ys r zs* **where**
      rec1: *bal* (*n div 2*) *xs* = (*l*, *ys*) **and**
      rec2: *bal* (*n* − *1* − *n div 2*) (*tl ys*) = (*r*, *zs*) **and**
      t: *t* = ⟨*l*, *hd ys*, *r*⟩
      by(*auto simp add*: *bal_simps split*: *prod.splits*)
   have *l*: *wbalanced l* **using** *less.IH*[*OF* _ _ *rec1*] *less.prems*(*1*) **by** *linarith*
    have *wbalanced r*
     using *rec1 rec2 bal_length*[*OF* _ *rec1*] *less.prems*(*1*) **by**(*intro less.IH*) *auto*
     with *l t bal_length*[*OF* _ *rec1*] *less.prems*(*1*) *bal_inorder*[*OF* _ *rec1*] *bal_inorder*[*OF* _ *rec2*]
    show *?thesis* **by** *auto*
  qed
qed

An alternative proof via *wbalanced ?t* ⟹ *acomplete ?t*:

lemma ⟦ *n* ≤ *length xs*; *bal n xs* = (*t*,*ys*) ⟧ ⟹ *acomplete t*
by(*rule acomplete_if_wbalanced*[*OF wbalanced_bal*])

lemma *wbalanced_bal_list*[*simp*]: $n \leq$ *length xs* $\Longrightarrow$ *wbalanced* (*bal_list n xs*)
by(*simp add*: *bal_list_def*) (*metis prod.collapse wbalanced_bal*)

lemma *wbalanced_balance_list*[*simp*]: *wbalanced* (*balance_list xs*)
by(*simp add*: *balance_list_def*)

lemma *wbalanced_bal_tree*[*simp*]: $n \leq$ *size t* $\Longrightarrow$ *wbalanced* (*bal_tree n t*)
by(*simp add*: *bal_tree_def*)

lemma *wbalanced_balance_tree*: *wbalanced* (*balance_tree t*)
by (*simp add*: *balance_tree_def*)

hide_const (**open**) *bal*

end

3 Three-Way Comparison

theory *Cmp*
imports *Main*
begin

datatype *cmp_val* = *LT* | *EQ* | *GT*

definition *cmp* :: *'a*:: *linorder* ⇒ *'a* ⇒ *cmp_val* **where**
cmp x y = (*if x* < *y then LT else if x*=*y then EQ else GT*)

lemma
  LT[*simp*]: *cmp x y* = *LT* ⟷ *x* < *y*
and *EQ*[*simp*]: *cmp x y* = *EQ* ⟷ *x* = *y*
and *GT*[*simp*]: *cmp x y* = *GT* ⟷ *x* > *y*
by (*auto simp*: *cmp_def*)

lemma *case_cmp_if*[*simp*]: (*case c of EQ* ⇒ *e* | *LT* ⇒ *l* | *GT* ⇒ *g*) =
  (*if c* = *LT then l else if c* = *GT then g else e*)
by(*simp split*: *cmp_val.split*)

end

4 Lists Sorted wrt <

theory *Sorted_Less*
imports *Less_False*
begin

hide_const *sorted*

Is a list sorted without duplicates, i.e., wrt <?.

abbreviation *sorted* :: *'a::linorder list* ⇒ *bool* **where**
sorted ≡ *sorted_wrt* (<)

lemmas *sorted_wrt_Cons* = *sorted_wrt.simps(2)*

The definition of *sorted_wrt* relates each element to all the elements after it. This causes a blowup of the formulas. Thus we simplify matters by only comparing adjacent elements.

declare
  sorted_wrt.simps(2)[simp del]
  sorted_wrt1[simp] sorted_wrt2[OF transp_on_less, simp]

lemma *sorted_cons*: *sorted* (*x#xs*) ⟹ *sorted xs*
by(*simp add*: *sorted_wrt_Cons*)

lemma *sorted_cons'*: *ASSUMPTION* (*sorted* (*x#xs*)) ⟹ *sorted xs*
by(*rule ASSUMPTION_D* [*THEN sorted_cons*])

lemma *sorted_snoc*: *sorted* (*xs* @ [*y*]) ⟹ *sorted xs*
by(*simp add*: *sorted_wrt_append*)

lemma *sorted_snoc'*: *ASSUMPTION* (*sorted* (*xs* @ [*y*])) ⟹ *sorted xs*
by(*rule ASSUMPTION_D* [*THEN sorted_snoc*])

lemma *sorted_mid_iff*:
  sorted(*xs* @ *y* # *ys*) = (*sorted*(*xs* @ [*y*]) ∧ *sorted*(*y* # *ys*))
by(*fastforce simp add*: *sorted_wrt_Cons sorted_wrt_append*)

lemma *sorted_mid_iff2*:
  sorted(*x* # *xs* @ *y* # *ys*) =
  (*sorted*(*x* # *xs*) ∧ *x* < *y* ∧ *sorted*(*xs* @ [*y*]) ∧ *sorted*(*y* # *ys*))
by(*fastforce simp add*: *sorted_wrt_Cons sorted_wrt_append*)

lemma *sorted_mid_iff'*: *NO_MATCH* [] *ys* ⟹
  sorted(*xs* @ *y* # *ys*) = (*sorted*(*xs* @ [*y*]) ∧ *sorted*(*y* # *ys*))

by(*rule sorted_mid_iff*)

lemmas *sorted_lems* = *sorted_mid_iff′ sorted_mid_iff2 sorted_cons′ sorted_snoc′*

Splay trees need two additional *sorted* lemmas:

lemma *sorted_snoc_le*:
  ASSUMPTION(*sorted*(*xs* @ [*x*])) $\Longrightarrow$ *x* $\leq$ *y* $\Longrightarrow$ *sorted* (*xs* @ [*y*])
by (*auto simp add*: *sorted_wrt_append ASSUMPTION_def*)

lemma *sorted_Cons_le*:
  ASSUMPTION(*sorted*(*x* # *xs*)) $\Longrightarrow$ *y* $\leq$ *x* $\Longrightarrow$ *sorted* (*y* # *xs*)
by (*auto simp add*: *sorted_wrt_Cons ASSUMPTION_def*)

end

5 List Insertion and Deletion

theory *List_Ins_Del*
imports *Sorted_Less*
begin

5.1 Elements in a list

lemma *sorted_Cons_iff*:
  sorted(*x* # *xs*) = (($\forall$ *y* $\in$ *set xs*. *x* < *y*) $\wedge$ *sorted xs*)
by(*simp add*: *sorted_wrt_Cons*)

lemma *sorted_snoc_iff*:
  sorted(*xs* @ [*x*]) = (*sorted xs* $\wedge$ ($\forall$ *y* $\in$ *set xs*. *y* < *x*))
by(*simp add*: *sorted_wrt_append*)

lemmas *isin_simps* = *sorted_mid_iff′ sorted_Cons_iff sorted_snoc_iff*

5.2 Inserting into an ordered list without duplicates:

fun *ins_list* :: *′a*::*linorder* ⇒ *′a list* ⇒ *′a list* **where**
ins_list x [] = [*x*] |
ins_list x (*a*#*xs*) =
  (*if x* < *a then x*#*a*#*xs else if x*=*a then a*#*xs else a* # *ins_list x xs*)

lemma *set_ins_list*: *set* (*ins_list x xs*) = *set xs* $\cup$ {*x*}
by(*induction xs*) *auto*

lemma *sorted_ins_list*: *sorted xs* $\Longrightarrow$ *sorted*(*ins_list x xs*)
by(*induction xs rule*: *induct_list012*) *auto*

lemma *ins_list_sorted*: *sorted* (*xs* @ [*a*]) $\Longrightarrow$
ins_list x (*xs* @ *a* # *ys*) =
(*if x* < *a then ins_list x xs* @ (*a*#*ys*) *else xs* @ *ins_list x* (*a*#*ys*))
by(*induction xs*) (*auto simp*: *sorted_lems*)

In principle, *sorted* (*?xs* @ [*?a*]) $\Longrightarrow$ *ins_list ?x* (*?xs* @ *?a* # *?ys*) = (*if ?x* < *?a then ins_list ?x ?xs* @ *?a* # *?ys else ?xs* @ *ins_list ?x* (*?a* # *?ys*)) suffices, but the following two corollaries speed up proofs.

corollary *ins_list_sorted1*: *sorted* (*xs* @ [*a*]) $\Longrightarrow$ *a* $\leq$ *x* $\Longrightarrow$
ins_list x (*xs* @ *a* # *ys*) = *xs* @ *ins_list x* (*a*#*ys*)
by(*auto simp add*: *ins_list_sorted*)

corollary *ins_list_sorted2*: *sorted* (*xs* @ [*a*]) $\Longrightarrow$ *x* < *a* $\Longrightarrow$
ins_list x (*xs* @ *a* # *ys*) = *ins_list x xs* @ (*a*#*ys*)
by(*auto simp*: *ins_list_sorted*)

lemmas *ins_list_simps* = *sorted_lems ins_list_sorted1 ins_list_sorted2*

Splay trees need two additional *ins_list* lemmas:

lemma *ins_list_Cons*: *sorted* (*x* # *xs*) $\Longrightarrow$ *ins_list x xs* = *x* # *xs*
by (*induction xs*) *auto*

lemma *ins_list_snoc*: *sorted* (*xs* @ [*x*]) $\Longrightarrow$ *ins_list x xs* = *xs* @ [*x*]
by(*induction xs*) (*auto simp add*: *sorted_mid_iff2*)

5.3 Delete one occurrence of an element from a list:

fun *del_list* :: *'a* $\Rightarrow$ *'a list* $\Rightarrow$ *'a list* **where**
del_list x [] = [] |
del_list x (*a*#*xs*) = (*if x*=*a then xs else a* # *del_list x xs*)

lemma *del_list_idem*: *x* $\notin$ *set xs* $\Longrightarrow$ *del_list x xs* = *xs*
by (*induct xs*) *simp_all*

lemma *set_del_list*:
sorted xs $\Longrightarrow$ *set* (*del_list x xs*) = *set xs* − {*x*}
by(*induct xs*) (*auto simp*: *sorted_Cons_iff*)

lemma *sorted_del_list*: *sorted xs* $\Longrightarrow$ *sorted*(*del_list x xs*)
apply(*induction xs rule*: *induct_list012*)
apply *auto*

by (*meson order.strict_trans sorted_Cons_iff*)

lemma *del_list_sorted*: *sorted* (*xs* @ *a* # *ys*) $\Longrightarrow$
del_list x (*xs* @ *a* # *ys*) = (*if x* < *a then del_list x xs* @ *a* # *ys else xs* @ *del_list x* (*a* # *ys*))
by(*induction xs*)
(*fastforce simp*: *sorted_lems sorted_Cons_iff intro*!: *del_list_idem*)+

In principle, *sorted* (*?xs* @ *?a* # *?ys*) $\Longrightarrow$ *del_list ?x* (*?xs* @ *?a* # *?ys*) = (*if ?x* < *?a then del_list ?x ?xs* @ *?a* # *?ys else ?xs* @ *del_list ?x* (*?a* # *?ys*)) suffices, but the following corollaries speed up proofs.

corollary *del_list_sorted1*: *sorted* (*xs* @ *a* # *ys*) $\Longrightarrow$ *a* $\le$ *x* $\Longrightarrow$
del_list x (*xs* @ *a* # *ys*) = *xs* @ *del_list x* (*a* # *ys*)
by (*auto simp*: *del_list_sorted*)

corollary *del_list_sorted2*: *sorted* (*xs* @ *a* # *ys*) $\Longrightarrow$ *x* < *a* $\Longrightarrow$
del_list x (*xs* @ *a* # *ys*) = *del_list x xs* @ *a* # *ys*
by (*auto simp*: *del_list_sorted*)

corollary *del_list_sorted3*:
sorted (*xs* @ *a* # *ys* @ *b* # *zs*) $\Longrightarrow$ *x* < *b* $\Longrightarrow$
del_list x (*xs* @ *a* # *ys* @ *b* # *zs*) = *del_list x* (*xs* @ *a* # *ys*) @ *b* # *zs*
by (*auto simp*: *del_list_sorted sorted_lems*)

corollary *del_list_sorted4*:
sorted (*xs* @ *a* # *ys* @ *b* # *zs* @ *c* # *us*) $\Longrightarrow$ *x* < *c* $\Longrightarrow$
del_list x (*xs* @ *a* # *ys* @ *b* # *zs* @ *c* # *us*) = *del_list x* (*xs* @ *a* # *ys* @ *b* # *zs*) @ *c* # *us*
by (*auto simp*: *del_list_sorted sorted_lems*)

corollary *del_list_sorted5*:
sorted (*xs* @ *a* # *ys* @ *b* # *zs* @ *c* # *us* @ *d* # *vs*) $\Longrightarrow$ *x* < *d* $\Longrightarrow$
del_list x (*xs* @ *a* # *ys* @ *b* # *zs* @ *c* # *us* @ *d* # *vs*) =
del_list x (*xs* @ *a* # *ys* @ *b* # *zs* @ *c* # *us*) @ *d* # *vs*
by (*auto simp*: *del_list_sorted sorted_lems*)

lemmas *del_list_simps* = *sorted_lems*
del_list_sorted1
del_list_sorted2
del_list_sorted3
del_list_sorted4
del_list_sorted5

Splay trees need two additional *del_list* lemmas:

lemma *del_list_notin_Cons*: *sorted* (*x* # *xs*) ⟹ *del_list x xs* = *xs*
by(*induction xs*)(*fastforce simp*: *sorted_Cons_iff*)+

lemma *del_list_sorted_app*:
  sorted(*xs* @ [*x*]) ⟹ *del_list x* (*xs* @ *ys*) = *xs* @ *del_list x ys*
by (*induction xs*) (*auto simp*: *sorted_mid_iff2*)

end

6 Specifications of Set ADT

theory *Set_Specs*
imports *List_Ins_Del*
begin

The basic set interface with traditional *set*-based specification:

locale *Set* =
fixes *empty* :: ′*s*
fixes *insert* :: ′*a* ⇒ ′*s* ⇒ ′*s*
fixes *delete* :: ′*a* ⇒ ′*s* ⇒ ′*s*
fixes *isin* :: ′*s* ⇒ ′*a* ⇒ *bool*
fixes *set* :: ′*s* ⇒ ′*a set*
fixes *invar* :: ′*s* ⇒ *bool*
assumes *set_empty*: *set empty* = {}
assumes *set_isin*: *invar s* ⟹ *isin s x* = (*x* ∈ *set s*)
assumes *set_insert*: *invar s* ⟹ *set*(*insert x s*) = *set s* ∪ {*x*}
assumes *set_delete*: *invar s* ⟹ *set*(*delete x s*) = *set s* − {*x*}
assumes *invar_empty*: *invar empty*
assumes *invar_insert*: *invar s* ⟹ *invar*(*insert x s*)
assumes *invar_delete*: *invar s* ⟹ *invar*(*delete x s*)

lemmas (**in** *Set*) *set_specs* =
  set_empty set_isin set_insert set_delete invar_empty invar_insert invar_delete

The basic set interface with *inorder*-based specification:

locale *Set_by_Ordered* =
fixes *empty* :: ′*t*
fixes *insert* :: ′*a*::*linorder* ⇒ ′*t* ⇒ ′*t*
fixes *delete* :: ′*a* ⇒ ′*t* ⇒ ′*t*
fixes *isin* :: ′*t* ⇒ ′*a* ⇒ *bool*
fixes *inorder* :: ′*t* ⇒ ′*a list*
fixes *inv* :: ′*t* ⇒ *bool*
assumes *inorder_empty*: *inorder empty* = []

assumes *isin*: *inv t* ∧ *sorted*(*inorder t*) ⟹
  isin t x = (*x* ∈ *set* (*inorder t*))
assumes *inorder_insert*: *inv t* ∧ *sorted*(*inorder t*) ⟹
  inorder(*insert x t*) = *ins_list x* (*inorder t*)
assumes *inorder_delete*: *inv t* ∧ *sorted*(*inorder t*) ⟹
  inorder(*delete x t*) = *del_list x* (*inorder t*)
assumes *inorder_inv_empty*: *inv empty*
assumes *inorder_inv_insert*: *inv t* ∧ *sorted*(*inorder t*) ⟹ *inv*(*insert x t*)
assumes *inorder_inv_delete*: *inv t* ∧ *sorted*(*inorder t*) ⟹ *inv*(*delete x t*)

begin

It implements the traditional specification:

definition *set* :: *'t* ⇒ *'a set* **where**
set = *List.set o inorder*

definition *invar* :: *'t* ⇒ *bool* **where**
invar t = (*inv t* ∧ *sorted* (*inorder t*))

sublocale *Set*
  empty insert delete isin set invar
proof(*standard*, *goal_cases*)
  case *1* **show** *?case* **by** (*auto simp*: *inorder_empty set_def*)
next
  case *2* **thus** *?case* **by**(*simp add*: *isin invar_def set_def*)
next
  case *3* **thus** *?case* **by**(*simp add*: *inorder_insert set_ins_list set_def in-var_def*)
next
  case (*4 s x*) **thus** *?case*
    by (*auto simp*: *inorder_delete set_del_list invar_def set_def*)
next
  case *5* **thus** *?case* **by**(*simp add*: *inorder_empty inorder_inv_empty in-var_def*)
next
  case *6* **thus** *?case* **by**(*simp add*: *inorder_insert inorder_inv_insert sorted_ins_list invar_def*)
next
  case *7* **thus** *?case* **by** (*auto simp*: *inorder_delete inorder_inv_delete sorted_del_list invar_def*)
qed

end

Set2 = Set with binary operations:

locale *Set2* = *Set*
  where *insert* = *insert* **for** *insert* :: $'a \Rightarrow 's \Rightarrow 's$ +
fixes *union* :: $'s \Rightarrow 's \Rightarrow 's$
fixes *inter* :: $'s \Rightarrow 's \Rightarrow 's$
fixes *diff* :: $'s \Rightarrow 's \Rightarrow 's$
assumes *set_union*: ⟦ *invar s1*; *invar s2* ⟧ ⟹ *set*(*union s1 s2*) = *set s1* ∪ *set s2*
assumes *set_inter*: ⟦ *invar s1*; *invar s2* ⟧ ⟹ *set*(*inter s1 s2*) = *set s1* ∩ *set s2*
assumes *set_diff*: ⟦ *invar s1*; *invar s2* ⟧ ⟹ *set*(*diff s1 s2*) = *set s1* − *set s2*
assumes *invar_union*: ⟦ *invar s1*; *invar s2* ⟧ ⟹ *invar*(*union s1 s2*)
assumes *invar_inter*: ⟦ *invar s1*; *invar s2* ⟧ ⟹ *invar*(*inter s1 s2*)
assumes *invar_diff*: ⟦ *invar s1*; *invar s2* ⟧ ⟹ *invar*(*diff s1 s2*)

end

7 Unbalanced Tree Implementation of Set

theory *Tree_Set*
imports
  HOL−Library.Tree
  Cmp
  Set_Specs
begin

definition *empty* :: *'a tree* **where**
empty = *Leaf*

fun *isin* :: *'a::linorder tree* ⇒ *'a* ⇒ *bool* **where**
isin Leaf x = *False* |
isin (*Node l a r*) *x* =
  (*case cmp x a of*
     LT ⇒ *isin l x* |
     EQ ⇒ *True* |
     GT ⇒ *isin r x*)

hide_const (**open**) *insert*

fun *insert* :: *'a::linorder* ⇒ *'a tree* ⇒ *'a tree* **where**
insert x Leaf = *Node Leaf x Leaf* |
insert x (*Node l a r*) =
  (*case cmp x a of*

$LT \Rightarrow Node\ (insert\ x\ l)\ a\ r\ |$
$EQ \Rightarrow Node\ l\ a\ r\ |$
$GT \Rightarrow Node\ l\ a\ (insert\ x\ r))$

Deletion by replacing:

fun *split_min* :: *'a tree* ⇒ *'a* ∗ *'a tree* **where**
split_min (*Node l a r*) =
 (*if l* = *Leaf then* (*a*,*r*) *else let* (*x*,*l'*) = *split_min l in* (*x*, *Node l' a r*))

fun *delete* :: *'a*::*linorder* ⇒ *'a tree* ⇒ *'a tree* **where**
delete x Leaf = *Leaf* |
delete x (*Node l a r*) =
 (*case cmp x a of*
  LT ⇒ *Node* (*delete x l*) *a r* |
  GT ⇒ *Node l a* (*delete x r*) |
  EQ ⇒ *if r* = *Leaf then l else let* (*a'*,*r'*) = *split_min r in Node l a' r'*)

Deletion by joining:

fun *join* :: (*'a*::*linorder*)*tree* ⇒ *'a tree* ⇒ *'a tree* **where**
join t Leaf = *t* |
join Leaf t = *t* |
join (*Node t1 a t2*) (*Node t3 b t4*) =
 (*case join t2 t3 of*
  Leaf ⇒ *Node t1 a* (*Node Leaf b t4*) |
  Node u2 x u3 ⇒ *Node* (*Node t1 a u2*) *x* (*Node u3 b t4*))

fun *delete2* :: *'a*::*linorder* ⇒ *'a tree* ⇒ *'a tree* **where**
delete2 x Leaf = *Leaf* |
delete2 x (*Node l a r*) =
 (*case cmp x a of*
  LT ⇒ *Node* (*delete2 x l*) *a r* |
  GT ⇒ *Node l a* (*delete2 x r*) |
  EQ ⇒ *join l r*)

7.1 Functional Correctness Proofs

lemma *isin_set*: *sorted*(*inorder t*) ⟹ *isin t x* = (*x* ∈ *set* (*inorder t*))
by (*induction t*) (*auto simp*: *isin_simps*)

lemma *inorder_insert*:
 sorted(*inorder t*) ⟹ *inorder*(*insert x t*) = *ins_list x* (*inorder t*)
by(*induction t*) (*auto simp*: *ins_list_simps*)

```
lemma split_minD:
  split_min t = (x,t') ⟹ t ≠ Leaf ⟹ x # inorder t' = inorder t
by(induction t arbitrary: t' rule: split_min.induct)
  (auto simp: sorted_lems split: prod.splits if_splits)

lemma inorder_delete:
  sorted(inorder t) ⟹ inorder(delete x t) = del_list x (inorder t)
by(induction t) (auto simp: del_list_simps split_minD split: prod.splits)

interpretation S: Set_by_Ordered
where empty = empty and isin = isin and insert = insert and delete =
delete
and inorder = inorder and inv = λ_. True
proof (standard, goal_cases)
  case 1 show ?case by (simp add: empty_def)
next
  case 2 thus ?case by(simp add: isin_set)
next
  case 3 thus ?case by(simp add: inorder_insert)
next
  case 4 thus ?case by(simp add: inorder_delete)
qed (rule TrueI)+

lemma inorder_join:
  inorder(join l r) = inorder l @ inorder r
by(induction l r rule: join.induct) (auto split: tree.split)

lemma inorder_delete2:
  sorted(inorder t) ⟹ inorder(delete2 x t) = del_list x (inorder t)
by(induction t) (auto simp: inorder_join del_list_simps)

interpretation S2: Set_by_Ordered
where empty = empty and isin = isin and insert = insert and delete =
delete2
and inorder = inorder and inv = λ_. True
proof (standard, goal_cases)
  case 1 show ?case by (simp add: empty_def)
next
  case 2 thus ?case by(simp add: isin_set)
next
  case 3 thus ?case by(simp add: inorder_insert)
next
  case 4 thus ?case by(simp add: inorder_delete2)
qed (rule TrueI)+
```

end

8 Association List Update and Deletion

theory *AList_Upd_Del*
imports *Sorted_Less*
begin

abbreviation *sorted1 ps* ≡ *sorted*(*map fst ps*)

Define own *map_of* function to avoid pulling in an unknown amount of lemmas implicitly (via the simpset).

hide_const (**open**) *map_of*

fun *map_of* :: ('*a*∗'*b*)*list* ⇒ '*a* ⇒ '*b option* **where**
map_of [] = (λ*x*. *None*) |
map_of ((*a*,*b*)#*ps*) = (λ*x*. *if x*=*a then Some b else map_of ps x*)

Updating an association list:

fun *upd_list* :: '*a*::*linorder* ⇒ '*b* ⇒ ('*a*∗'*b*) *list* ⇒ ('*a*∗'*b*) *list* **where**
upd_list x y [] = [(*x*,*y*)] |
upd_list x y ((*a*,*b*)#*ps*) =
  (*if x* < *a then* (*x*,*y*)#(*a*,*b*)#*ps else*
  if x = *a then* (*x*,*y*)#*ps else* (*a*,*b*) # *upd_list x y ps*)

fun *del_list* :: '*a*::*linorder* ⇒ ('*a*∗'*b*)*list* ⇒ ('*a*∗'*b*)*list* **where**
del_list x [] = [] |
del_list x ((*a*,*b*)#*ps*) = (*if x* = *a then ps else* (*a*,*b*) # *del_list x ps*)

8.1 Lemmas for *map_of*

lemma *map_of_ins_list*: *map_of* (*upd_list x y ps*) = (*map_of ps*)(*x* := *Some y*)
by(*induction ps*) *auto*

lemma *map_of_append*: *map_of* (*ps* @ *qs*) *x* =
  (*case map_of ps x of None* ⇒ *map_of qs x* | *Some y* ⇒ *Some y*)
by(*induction ps*)(*auto*)

lemma *map_of_None*: *sorted* (*x* # *map fst ps*) ⟹ *map_of ps x* = *None*
by (*induction ps*) (*fastforce simp*: *sorted_lems sorted_wrt_Cons*)+

lemma *map_of_None2*: *sorted* (*map fst ps* @ [*x*]) ⟹ *map_of ps x* = *None*
by (*induction ps*) (*auto simp*: *sorted_lems*)

lemma *map_of_del_list*: *sorted1 ps* ⟹
map_of(*del_list x ps*) = (*map_of ps*)(*x* := *None*)
by(*induction ps*) (*auto simp*: *map_of_None sorted_lems fun_eq_iff*)

lemma *map_of_sorted_Cons*: *sorted* (*a* # *map fst ps*) ⟹ *x* < *a* ⟹
map_of ps x = *None*
by (*simp add*: *map_of_None sorted_Cons_le*)

lemma *map_of_sorted_snoc*: *sorted* (*map fst ps* @ [*a*]) ⟹ *a* ≤ *x* ⟹
map_of ps x = *None*
by (*simp add*: *map_of_None2 sorted_snoc_le*)

lemmas *map_of_sorteds* = *map_of_sorted_Cons map_of_sorted_snoc*
lemmas *map_of_simps* = *sorted_lems map_of_append map_of_sorteds*

8.2 Lemmas for *upd_list*

lemma *sorted_upd_list*: *sorted1 ps* ⟹ *sorted1* (*upd_list x y ps*)
apply(*induction ps*)
 apply *simp*
apply(*case_tac ps*)
 apply *auto*
done

lemma *upd_list_sorted*: *sorted1* (*ps* @ [(*a*,*b*)]) ⟹
upd_list x y (*ps* @ (*a*,*b*) # *qs*) =
(*if x* < *a then upd_list x y ps* @ (*a*,*b*) # *qs*
else ps @ *upd_list x y* ((*a*,*b*) # *qs*))
by(*induction ps*) (*auto simp*: *sorted_lems*)

In principle, *sorted1* (*?ps* @ [(*?a*, *?b*)]) ⟹ *upd_list ?x ?y* (*?ps* @ (*?a*, *?b*) # *?qs*) = (*if ?x* < *?a then upd_list ?x ?y ?ps* @ (*?a*, *?b*) # *?qs else ?ps* @ *upd_list ?x ?y* ((*?a*, *?b*) # *?qs*)) suffices, but the following two corollaries speed up proofs.

corollary *upd_list_sorted1*: ⟦ *sorted* (*map fst ps* @ [*a*]); *x* < *a* ⟧ ⟹
upd_list x y (*ps* @ (*a*,*b*) # *qs*) = *upd_list x y ps* @ (*a*,*b*) # *qs*
by (*auto simp*: *upd_list_sorted*)

corollary *upd_list_sorted2*: ⟦ *sorted* (*map fst ps* @ [*a*]); *a* ≤ *x* ⟧ ⟹
upd_list x y (*ps* @ (*a*,*b*) # *qs*) = *ps* @ *upd_list x y* ((*a*,*b*) # *qs*)

by (*auto simp*: *upd_list_sorted*)

lemmas *upd_list_simps* = *sorted_lems upd_list_sorted1 upd_list_sorted2*

Splay trees need two additional *upd_list* lemmas:

lemma *upd_list_Cons*:
sorted1 ((*x*,*y*) # *xs*) ⟹ *upd_list x y xs* = (*x*,*y*) # *xs*
by (*induction xs*) *auto*

lemma *upd_list_snoc*:
sorted1 (*xs* @ [(*x*,*y*)]) ⟹ *upd_list x y xs* = *xs* @ [(*x*,*y*)]
by(*induction xs*) (*auto simp add*: *sorted_mid_iff2*)

8.3 Lemmas for *del_list*

lemma *sorted_del_list*: *sorted1 ps* ⟹ *sorted1* (*del_list x ps*)
apply(*induction ps*)
apply *simp*
apply(*case_tac ps*)
apply (*auto simp*: *sorted_Cons_le*)
done

lemma *del_list_idem*: *x* ∉ *set*(*map fst xs*) ⟹ *del_list x xs* = *xs*
by (*induct xs*) *auto*

lemma *del_list_sorted*: *sorted1* (*ps* @ (*a*,*b*) # *qs*) ⟹
del_list x (*ps* @ (*a*,*b*) # *qs*) =
(*if x* < *a then del_list x ps* @ (*a*,*b*) # *qs*
else ps @ *del_list x* ((*a*,*b*) # *qs*))
by(*induction ps*)
(*fastforce simp*: *sorted_lems sorted_wrt_Cons intro*!: *del_list_idem*)+

In principle, *sorted1* (*?ps* @ (*?a*, *?b*) # *?qs*) ⟹ *del_list ?x* (*?ps* @ (*?a*, *?b*) # *?qs*) = (*if ?x* < *?a then del_list ?x ?ps* @ (*?a*, *?b*) # *?qs else ?ps* @ *del_list ?x* ((*?a*, *?b*) # *?qs*)) suffices, but the following corollaries speed up proofs.

corollary *del_list_sorted1*: *sorted1* (*xs* @ (*a*,*b*) # *ys*) ⟹ *a* ≤ *x* ⟹
del_list x (*xs* @ (*a*,*b*) # *ys*) = *xs* @ *del_list x* ((*a*,*b*) # *ys*)
by (*auto simp*: *del_list_sorted*)

lemma *del_list_sorted2*: *sorted1* (*xs* @ (*a*,*b*) # *ys*) ⟹ *x* < *a* ⟹
del_list x (*xs* @ (*a*,*b*) # *ys*) = *del_list x xs* @ (*a*,*b*) # *ys*
by (*auto simp*: *del_list_sorted*)

lemma *del_list_sorted3*:
sorted1 (*xs* @ (*a*,*a*′) # *ys* @ (*b*,*b*′) # *zs*) ⟹ *x* < *b* ⟹
del_list x (*xs* @ (*a*,*a*′) # *ys* @ (*b*,*b*′) # *zs*) = *del_list x* (*xs* @ (*a*,*a*′) # *ys*) @ (*b*,*b*′) # *zs*
by (*auto simp*: *del_list_sorted sorted_lems*)

lemma *del_list_sorted4*:
sorted1 (*xs* @ (*a*,*a*′) # *ys* @ (*b*,*b*′) # *zs* @ (*c*,*c*′) # *us*) ⟹ *x* < *c* ⟹
del_list x (*xs* @ (*a*,*a*′) # *ys* @ (*b*,*b*′) # *zs* @ (*c*,*c*′) # *us*) = *del_list x* (*xs* @ (*a*,*a*′) # *ys* @ (*b*,*b*′) # *zs*) @ (*c*,*c*′) # *us*
by (*auto simp*: *del_list_sorted sorted_lems*)

lemma *del_list_sorted5*:
sorted1 (*xs* @ (*a*,*a*′) # *ys* @ (*b*,*b*′) # *zs* @ (*c*,*c*′) # *us* @ (*d*,*d*′) # *vs*) ⟹ *x* < *d* ⟹
del_list x (*xs* @ (*a*,*a*′) # *ys* @ (*b*,*b*′) # *zs* @ (*c*,*c*′) # *us* @ (*d*,*d*′) # *vs*) =
del_list x (*xs* @ (*a*,*a*′) # *ys* @ (*b*,*b*′) # *zs* @ (*c*,*c*′) # *us*) @ (*d*,*d*′) # *vs*
by (*auto simp*: *del_list_sorted sorted_lems*)

lemmas *del_list_simps* = *sorted_lems*
del_list_sorted1
del_list_sorted2
del_list_sorted3
del_list_sorted4
del_list_sorted5

Splay trees need two additional *del_list* lemmas:

lemma *del_list_notin_Cons*: *sorted* (*x* # *map fst xs*) ⟹ *del_list x xs* = *xs*
by(*induction xs*)(*fastforce simp*: *sorted_wrt_Cons*)+

lemma *del_list_sorted_app*:
sorted(*map fst xs* @ [*x*]) ⟹ *del_list x* (*xs* @ *ys*) = *xs* @ *del_list x ys*
by (*induction xs*) (*auto simp*: *sorted_mid_iff2*)

end

9 Specifications of Map ADT

theory *Map_Specs*
imports *AList_Upd_Del*
begin

The basic map interface with $'a \Rightarrow 'b$ *option* based specification:

locale *Map* =
fixes *empty* :: $'m$
fixes *update* :: $'a \Rightarrow 'b \Rightarrow 'm \Rightarrow 'm$
fixes *delete* :: $'a \Rightarrow 'm \Rightarrow 'm$
fixes *lookup* :: $'m \Rightarrow 'a \Rightarrow 'b$ *option*
fixes *invar* :: $'m \Rightarrow$ *bool*
assumes *map_empty*: *lookup empty* = (λ_. *None*)
and *map_update*: *invar m* ⟹ *lookup*(*update a b m*) = (*lookup m*)(*a* := *Some b*)
and *map_delete*: *invar m* ⟹ *lookup*(*delete a m*) = (*lookup m*)(*a* := *None*)
and *invar_empty*: *invar empty*
and *invar_update*: *invar m* ⟹ *invar*(*update a b m*)
and *invar_delete*: *invar m* ⟹ *invar*(*delete a m*)

lemmas (**in** *Map*) *map_specs* =
map_empty map_update map_delete invar_empty invar_update invar_delete

The basic map interface with *inorder*-based specification:

locale *Map_by_Ordered* =
fixes *empty* :: $'t$
fixes *update* :: $'a$::*linorder* $\Rightarrow 'b \Rightarrow 't \Rightarrow 't$
fixes *delete* :: $'a \Rightarrow 't \Rightarrow 't$
fixes *lookup* :: $'t \Rightarrow 'a \Rightarrow 'b$ *option*
fixes *inorder* :: $'t \Rightarrow ('a * 'b)$ *list*
fixes *inv* :: $'t \Rightarrow$ *bool*
assumes *inorder_empty*: *inorder empty* = []
and *inorder_lookup*: *inv t* ∧ *sorted1* (*inorder t*) ⟹
lookup t a = *map_of* (*inorder t*) *a*
and *inorder_update*: *inv t* ∧ *sorted1* (*inorder t*) ⟹
inorder(*update a b t*) = *upd_list a b* (*inorder t*)
and *inorder_delete*: *inv t* ∧ *sorted1* (*inorder t*) ⟹
inorder(*delete a t*) = *del_list a* (*inorder t*)
and *inorder_inv_empty*: *inv empty*
and *inorder_inv_update*: *inv t* ∧ *sorted1* (*inorder t*) ⟹ *inv*(*update a b t*)
and *inorder_inv_delete*: *inv t* ∧ *sorted1* (*inorder t*) ⟹ *inv*(*delete a t*)

begin

It implements the traditional specification:

definition *invar* :: $'t \Rightarrow$ *bool* **where**
invar t == *inv t* ∧ *sorted1* (*inorder t*)

sublocale *Map*

```
  empty update delete lookup invar
proof(standard, goal_cases)
  case 1 show ?case by (auto simp: inorder_lookup inorder_empty inorder_inv_empty)
next
  case 2 thus ?case
    by(simp add: fun_eq_iff inorder_update inorder_inv_update map_of_ins_list inorder_lookup
         sorted_upd_list invar_def)
next
  case 3 thus ?case
    by(simp add: fun_eq_iff inorder_delete inorder_inv_delete map_of_del_list inorder_lookup
         sorted_del_list invar_def)
next
  case 4 thus ?case by(simp add: inorder_empty inorder_inv_empty invar_def)
next
  case 5 thus ?case by(simp add: inorder_update inorder_inv_update sorted_upd_list invar_def)
next
  case 6 thus ?case by (auto simp: inorder_delete inorder_inv_delete sorted_del_list invar_def)
qed

end

end
```

10 Unbalanced Tree Implementation of Map

```
theory Tree_Map
imports
  Tree_Set
  Map_Specs
begin

fun lookup :: ('a::linorder*'b) tree ⇒ 'a ⇒ 'b option where
lookup Leaf x = None |
lookup (Node l (a,b) r) x =
  (case cmp x a of LT ⇒ lookup l x | GT ⇒ lookup r x | EQ ⇒ Some b)

fun update :: 'a::linorder ⇒ 'b ⇒ ('a*'b) tree ⇒ ('a*'b) tree where
```

update x y Leaf = *Node Leaf* (*x*,*y*) *Leaf* |
update x y (*Node l* (*a*,*b*) *r*) = (*case cmp x a of*
  LT ⇒ *Node* (*update x y l*) (*a*,*b*) *r* |
  EQ ⇒ *Node l* (*x*,*y*) *r* |
  GT ⇒ *Node l* (*a*,*b*) (*update x y r*))

fun *delete* :: *'a*::*linorder* ⇒ (*'a*∗*'b*) *tree* ⇒ (*'a*∗*'b*) *tree* **where**
delete x Leaf = *Leaf* |
delete x (*Node l* (*a*,*b*) *r*) = (*case cmp x a of*
  LT ⇒ *Node* (*delete x l*) (*a*,*b*) *r* |
  GT ⇒ *Node l* (*a*,*b*) (*delete x r*) |
  EQ ⇒ *if r* = *Leaf then l else let* (*ab'*,*r'*) = *split_min r in Node l ab' r'*)

10.1 Functional Correctness Proofs

lemma *lookup_map_of*:
  sorted1(*inorder t*) ⟹ *lookup t x* = *map_of* (*inorder t*) *x*
by (*induction t*) (*auto simp*: *map_of_simps split*: *option.split*)

lemma *inorder_update*:
  sorted1(*inorder t*) ⟹ *inorder*(*update a b t*) = *upd_list a b* (*inorder t*)
by(*induction t*) (*auto simp*: *upd_list_simps*)

lemma *inorder_delete*:
  sorted1(*inorder t*) ⟹ *inorder*(*delete x t*) = *del_list x* (*inorder t*)
by(*induction t*) (*auto simp*: *del_list_simps split_minD split*: *prod.splits*)

interpretation *M*: *Map_by_Ordered*
where *empty* = *empty* **and** *lookup* = *lookup* **and** *update* = *update* **and**
delete = *delete*
and *inorder* = *inorder* **and** *inv* = λ_. *True*
proof (*standard*, *goal_cases*)
  case *1* **show** *?case* **by** (*simp add*: *empty_def*)
next
  case *2* **thus** *?case* **by**(*simp add*: *lookup_map_of*)
next
  case *3* **thus** *?case* **by**(*simp add*: *inorder_update*)
next
  case *4* **thus** *?case* **by**(*simp add*: *inorder_delete*)
qed *auto*

end

11 Tree Rotations

theory *Tree_Rotations*
imports *HOL−Library.Tree*
begin

How to transform a tree into a list and into any other tree (with the same *inorder*) by rotations.

fun *is_list* :: *'a tree* ⇒ *bool* **where**
is_list (*Node l _ r*) = (*l* = *Leaf* ∧ *is_list r*) |
is_list Leaf = *True*

Termination proof via measure function. NB *size t* − *rlen t* works for the actual rotation equation but not for the second equation.

fun *rlen* :: *'a tree* ⇒ *nat* **where**
rlen Leaf = *0* |
rlen (*Node l x r*) = *rlen r* + *1*

lemma *rlen_le_size*: *rlen t* ≤ *size t*
by(*induction t*) *auto*

11.1 Without positions

function (*sequential*) *list_of* :: *'a tree* ⇒ *'a tree* **where**
list_of (*Node* (*Node A a B*) *b C*) = *list_of* (*Node A a* (*Node B b C*)) |
list_of (*Node Leaf a A*) = *Node Leaf a* (*list_of A*) |
list_of Leaf = *Leaf*
by *pat_completeness auto*

termination
proof
  let *?R* = *measure*(λ*t*. *2*∗*size t* − *rlen t*)
  show *wf ?R* **by** (*auto simp add*: *mlex_prod_def*)

  fix *A a B b C*
  show (*Node A a* (*Node B b C*), *Node* (*Node A a B*) *b C*) ∈ *?R*
    using *rlen_le_size*[*of C*] **by**(*simp*)

  fix *a A* **show** (*A*, *Node Leaf a A*) ∈ *?R* **using** *rlen_le_size*[*of A*] **by**(*simp*)
qed

lemma *is_list_rot*: *is_list*(*list_of t*)
by (*induction t rule*: *list_of.induct*) *auto*

lemma *inorder_rot*: *inorder*(*list_of t*) = *inorder t*

by (*induction t rule*: *list_of.induct*) *auto*

11.2 With positions

datatype *dir* = *L* | *R*

type_synonym *pos* = *dir list*

function (*sequential*) *rotR_poss* :: *'a tree* ⇒ *pos list* **where**
rotR_poss (*Node* (*Node A a B*) *b C*) = [] # *rotR_poss* (*Node A a* (*Node B b C*)) |
rotR_poss (*Node Leaf a A*) = *map* (*Cons R*) (*rotR_poss A*) |
rotR_poss Leaf = []
by *pat_completeness auto*

termination
proof
  let *?R* = *measure*(λ*t*. *2*∗*size t* − *rlen t*)
  show *wf ?R* **by** (*auto simp add*: *mlex_prod_def*)

  fix *A a B b C*
  show (*Node A a* (*Node B b C*), *Node* (*Node A a B*) *b C*) ∈ *?R*
    using *rlen_le_size*[*of C*] **by**(*simp*)

  fix *a A* **show** (*A*, *Node Leaf a A*) ∈ *?R* **using** *rlen_le_size*[*of A*] **by**(*simp*)
qed

fun *rotR* :: *'a tree* ⇒ *'a tree* **where**
rotR (*Node* (*Node A a B*) *b C*) = *Node A a* (*Node B b C*)

fun *rotL* :: *'a tree* ⇒ *'a tree* **where**
rotL (*Node A a* (*Node B b C*)) = *Node* (*Node A a B*) *b C*

fun *apply_at* :: (*'a tree* ⇒ *'a tree*) ⇒ *pos* ⇒ *'a tree* ⇒ *'a tree* **where**
  apply_at f [] *t* = *f t*
| *apply_at f* (*L* # *ds*) (*Node l a r*) = *Node* (*apply_at f ds l*) *a r*
| *apply_at f* (*R* # *ds*) (*Node l a r*) = *Node l a* (*apply_at f ds r*)

fun *apply_ats* :: (*'a tree* ⇒ *'a tree*) ⇒ *pos list* ⇒ *'a tree* ⇒ *'a tree* **where**
apply_ats _ [] *t* = *t* |
apply_ats f (*p*#*ps*) *t* = *apply_ats f ps* (*apply_at f p t*)

lemma *apply_ats_append*:
  apply_ats f (ps_1 @ ps_2) *t* = *apply_ats f* ps_2 (*apply_ats f* ps_1 *t*)

by (*induction* ps_1 *arbitrary*: *t*) *auto*

abbreviation *rotRs* ≡ *apply_ats rotR*
abbreviation *rotLs* ≡ *apply_ats rotL*

lemma *apply_ats_map_R*: *apply_ats f* (*map* ((#) *R*) *ps*) ⟨*l*, *a*, *r*⟩ = *Node l a* (*apply_ats f ps r*)
by(*induction ps arbitrary*: *r*) *auto*

lemma *inorder_rotRs_poss*: *inorder* (*rotRs* (*rotR_poss t*) *t*) = *inorder t*
apply(*induction t rule*: *rotR_poss.induct*)
apply(*auto simp*: *apply_ats_map_R*)
done

lemma *is_list_rotRs*: *is_list* (*rotRs* (*rotR_poss t*) *t*)
apply(*induction t rule*: *rotR_poss.induct*)
apply(*auto simp*: *apply_ats_map_R*)
done

lemma *is_list* (*rotRs ps t*) ⟶ *length ps* ≤ *length*(*rotR_poss t*)
quickcheck[*expect*=*counterexample*]
oops

lemma *length_rotRs_poss*: *length* (*rotR_poss t*) = *size t* − *rlen t*
proof(*induction t rule*: *rotR_poss.induct*)
  case (*1 A a B b C*)
  then show *?case* **using** *rlen_le_size*[*of C*] **by** *simp*
qed *auto*

lemma *is_list_inorder_same*:
  is_list t1 ⟹ *is_list t2* ⟹ *inorder t1* = *inorder t2* ⟹ *t1* = *t2*
proof(*induction t1 arbitrary*: *t2*)
  case *Leaf*
  then show *?case* **by** *simp*
next
  case *Node*
  then show *?case* **by** (*cases t2*) *simp_all*
qed

lemma *rot_id*: *rotLs* (*rev* (*rotR_poss t*)) (*rotRs* (*rotR_poss t*) *t*) = *t*
apply(*induction t rule*: *rotR_poss.induct*)
apply(*auto simp*: *apply_ats_map_R rev_map apply_ats_append*)
done

corollary *tree_to_tree_rotations*: **assumes** *inorder t1* = *inorder t2*
shows *rotLs* (*rev* (*rotR_poss t2*)) (*rotRs* (*rotR_poss t1*) *t1*) = *t2*
proof –
  have *rotRs* (*rotR_poss t1*) *t1* = *rotRs* (*rotR_poss t2*) *t2* (**is** *?L* = *?R*)
   by (*simp add*: *assms inorder_rotRs_poss is_list_inorder_same is_list_rotRs*)
  hence *rotLs* (*rev* (*rotR_poss t2*)) *?L* = *rotLs* (*rev* (*rotR_poss t2*)) *?R*
    by *simp*
  also have ... = *t2* **by**(*rule rot_id*)
  finally show *?thesis* **.**
qed

lemma *size_rlen_better_ub*: *size t* − *rlen t* ≤ *size t* − *1*
by (*cases t*) *auto*

end

12 Augmented Tree (Tree2)

theory *Tree2*
imports *HOL−Library.Tree*
begin

This theory provides the basic infrastructure for the type (*'a* × *'b*) *tree* of augmented trees where *'a* is the key and *'b* some additional information.

IMPORTANT: Inductions and cases analyses on augmented trees need to use the following two rules explicitly. They generate nodes of the form ⟨*l*, (*a*, *b*), *r*⟩ rather than ⟨*l*, *a*, *r*⟩ for trees of type *'a tree*.

lemmas *tree2_induct* = *tree.induct*[**where** *'a* = *'a* ∗ *'b*, *split_format*(*complete*)]

lemmas *tree2_cases* = *tree.exhaust*[**where** *'a* = *'a* ∗ *'b*, *split_format*(*complete*)]

fun *inorder* :: (*'a*∗*'b*)*tree* ⇒ *'a list* **where**
inorder Leaf = [] |
inorder (*Node l* (*a*,_) *r*) = *inorder l* @ *a* # *inorder r*

fun *set_tree* :: (*'a*∗*'b*) *tree* ⇒ *'a set* **where**
set_tree Leaf = {} |
set_tree (*Node l* (*a*,_) *r*) = {*a*} ∪ *set_tree l* ∪ *set_tree r*

fun *bst* :: (*'a*::*linorder*∗*'b*) *tree* ⇒ *bool* **where**
bst Leaf = *True* |
bst (*Node l* (*a*, _) *r*) = ((∀ *x* ∈ *set_tree l*. *x* < *a*) ∧ (∀ *x* ∈ *set_tree r*. *a* < *x*) ∧ *bst l* ∧ *bst r*)

lemma *finite_set_tree*[*simp*]: *finite*(*set_tree t*)
by(*induction t*) *auto*

lemma *eq_set_tree_empty*[*simp*]: *set_tree t* = {} ⟷ *t* = *Leaf*
by (*cases t*) *auto*

lemma *set_inorder*[*simp*]: *set* (*inorder t*) = *set_tree t*
by (*induction t*) *auto*

lemma *length_inorder*[*simp*]: *length* (*inorder t*) = *size t*
by (*induction t*) *auto*

end

13 Function *isin* for Tree2

theory *Isin2*
imports
  Tree2
  Cmp
  Set_Specs
begin

fun *isin* :: (′*a*::*linorder*∗′*b*) *tree* ⇒ ′*a* ⇒ *bool* **where**
isin Leaf x = *False* |
isin (*Node l* (*a*,_) *r*) *x* =
  (*case cmp x a of*
     LT ⇒ *isin l x* |
     EQ ⇒ *True* |
     GT ⇒ *isin r x*)

lemma *isin_set_inorder*: *sorted*(*inorder t*) ⟹ *isin t x* = (*x* ∈ *set*(*inorder* *t*))
by (*induction t rule*: *tree2_induct*) (*auto simp*: *isin_simps*)

lemma *isin_set_tree*: *bst t* ⟹ *isin t x* ⟷ *x* ∈ *set_tree t*
by(*induction t rule*: *tree2_induct*) *auto*

end

14 Interval Trees

theory *Interval_Tree*
imports
HOL−Data_Structures.Cmp
HOL−Data_Structures.List_Ins_Del
HOL−Data_Structures.Isin2
HOL−Data_Structures.Set_Specs
begin

14.1 Intervals

The following definition of intervals uses the **typedef** command to define the type of non-empty intervals as a subset of the type of pairs p where *fst* $p \leq$ *snd* p:

typedef (**overloaded**) $'a$::*linorder ivl* =
$\{p :: 'a \times 'a.$ *fst* $p \leq$ *snd* $p\}$ **by** *auto*

More precisely, $'a$ *ivl* is isomorphic with that subset via the function *Rep_ivl*. Hence the basic interval properties are not immediate but need simple proofs:

definition *low* :: $'a$::*linorder ivl* $\Rightarrow$ $'a$ **where**
low p = *fst* (*Rep_ivl* p)

definition *high* :: $'a$::*linorder ivl* $\Rightarrow$ $'a$ **where**
high p = *snd* (*Rep_ivl* p)

lemma *ivl_is_interval*: *low* $p \leq$ *high* p
by (*metis Rep_ivl high_def low_def mem_Collect_eq*)

lemma *ivl_inj*: *low* p = *low* $q \Longrightarrow$ *high* p = *high* $q \Longrightarrow p = q$
by (*metis Rep_ivl_inverse high_def low_def prod_eqI*)

Now we can forget how exactly intervals were defined.

instantiation *ivl* :: (*linorder*) *linorder* **begin**

definition *ivl_less*: $(x < y)$ = (*low* $x <$ *low* y | (*low* x = *low* $y \wedge$ *high* $x <$ *high* y))
definition *ivl_less_eq*: $(x \leq y)$ = (*low* $x <$ *low* y | (*low* x = *low* $y \wedge$ *high* $x \leq$ *high* y))

instance proof
fix $x\ y\ z$ :: $'a$ *ivl*
show a: $(x < y) = (x \leq y \wedge \neg\, y \leq x)$

using *ivl_less ivl_less_eq* **by** *force*
show *b*: $x \leq x$
by (*simp add*: *ivl_less_eq*)
show *c*: $x \leq y \Longrightarrow y \leq z \Longrightarrow x \leq z$
using *ivl_less_eq* **by** *fastforce*
show *d*: $x \leq y \Longrightarrow y \leq x \Longrightarrow x = y$
using *ivl_less_eq a ivl_inj ivl_less* **by** *fastforce*
show *e*: $x \leq y \vee y \leq x$
by (*meson ivl_less_eq leI not_less_iff_gr_or_eq*)
qed end

definition *overlap* :: ('*a*::*linorder*) *ivl* ⇒ '*a ivl* ⇒ *bool* **where**
overlap x y ⟷ (*high x* ≥ *low y* ∧ *high y* ≥ *low x*)

definition *has_overlap* :: ('*a*::*linorder*) *ivl set* ⇒ '*a ivl* ⇒ *bool* **where**
has_overlap S y ⟷ (∃ *x*∈*S*. *overlap x y*)

14.2 Interval Trees

type_synonym '*a ivl_tree* = ('*a ivl* ∗ '*a*) *tree*

fun *max_hi* :: ('*a*::*order_bot*) *ivl_tree* ⇒ '*a* **where**
max_hi Leaf = *bot* |
max_hi (*Node* _ (_,*m*) _) = *m*

definition *max3* :: ('*a*::*linorder*) *ivl* ⇒ '*a* ⇒ '*a* ⇒ '*a* **where**
max3 a m n = *max* (*high a*) (*max m n*)

fun *inv_max_hi* :: ('*a*::{*linorder*,*order_bot*}) *ivl_tree* ⇒ *bool* **where**
inv_max_hi Leaf ⟷ *True* |
inv_max_hi (*Node l* (*a*, *m*) *r*) ⟷ (*m* = *max3 a* (*max_hi l*) (*max_hi r*)
∧ *inv_max_hi l* ∧ *inv_max_hi r*)

lemma *max_hi_is_max*:
inv_max_hi t ⟹ *a* ∈ *set_tree t* ⟹ *high a* ≤ *max_hi t*
by (*induct t*, *auto simp add*: *max3_def max_def*)

lemma *max_hi_exists*:
inv_max_hi t ⟹ *t* ≠ *Leaf* ⟹ ∃ *a*∈*set_tree t*. *high a* = *max_hi t*
proof (*induction t rule*: *tree2_induct*)
case *Leaf*
then show *?case* **by** *auto*
next

```
  case N: (Node l v m r)
  then show ?case
  proof (cases l rule: tree2_cases)
    case Leaf
    then show ?thesis
      using N.prems(1) N.IH(2) by (cases r, auto simp add: max3_def
max_def le_bot)
  next
    case Nl: Node
    then show ?thesis
    proof (cases r rule: tree2_cases)
      case Leaf
      then show ?thesis
      using N.prems(1) N.IH(1) Nl by (auto simp add: max3_def max_def
le_bot)
    next
      case Nr: Node
      obtain p1 where p1: p1 ∈ set_tree l high p1 = max_hi l
        using N.IH(1) N.prems(1) Nl by auto
      obtain p2 where p2: p2 ∈ set_tree r high p2 = max_hi r
        using N.IH(2) N.prems(1) Nr by auto
      then show ?thesis
        using p1 p2 N.prems(1) by (auto simp add: max3_def max_def)
    qed
  qed
qed
```

14.3 Insertion and Deletion

```
definition node where
[simp]: node l a r = Node l (a, max3 a (max_hi l) (max_hi r)) r

fun insert :: 'a::{linorder,order_bot} ivl ⇒ 'a ivl_tree ⇒ 'a ivl_tree where
insert x Leaf = Node Leaf (x, high x) Leaf |
insert x (Node l (a, m) r) =
  (case cmp x a of
     EQ ⇒ Node l (a, m) r |
     LT ⇒ node (insert x l) a r |
     GT ⇒ node l a (insert x r))

lemma inorder_insert:
  sorted (inorder t) ⟹ inorder (insert x t) = ins_list x (inorder t)
by (induct t rule: tree2_induct) (auto simp: ins_list_simps)
```

lemma *inv_max_hi_insert*:
  inv_max_hi t ⟹ *inv_max_hi* (*insert x t*)
by (*induct t rule*: *tree2_induct*) (*auto simp add*: *max3_def*)

fun *split_min* :: *'a*::{*linorder*,*order_bot*} *ivl_tree* ⇒ *'a ivl* × *'a ivl_tree*
where
split_min (*Node l* (*a*, *m*) *r*) =
  (*if l* = *Leaf then* (*a*, *r*)
   else let (*x*,*l'*) = *split_min l in* (*x*, *node l' a r*))

fun *delete* :: *'a*::{*linorder*,*order_bot*} *ivl* ⇒ *'a ivl_tree* ⇒ *'a ivl_tree* **where**
delete x Leaf = *Leaf* |
delete x (*Node l* (*a*, *m*) *r*) =
  (*case cmp x a of*
     LT ⇒ *node* (*delete x l*) *a r* |
     GT ⇒ *node l a* (*delete x r*) |
     EQ ⇒ *if r* = *Leaf then l else*
           let (*a'*, *r'*) = *split_min r in node l a' r'*)

lemma *split_minD*:
  split_min t = (*x*,*t'*) ⟹ *t* ≠ *Leaf* ⟹ *x* # *inorder t'* = *inorder t*
by (*induct t arbitrary*: *t' rule*: *split_min.induct*)
   (*auto simp*: *sorted_lems split*: *prod.splits if_splits*)

lemma *inorder_delete*:
  sorted (*inorder t*) ⟹ *inorder* (*delete x t*) = *del_list x* (*inorder t*)
by (*induct t*)
   (*auto simp*: *del_list_simps split_minD Let_def split*: *prod.splits*)

lemma *inv_max_hi_split_min*:
  ⟦ *t* ≠ *Leaf*; *inv_max_hi t* ⟧ ⟹ *inv_max_hi* (*snd* (*split_min t*))
by (*induct t*) (*auto split*: *prod.splits*)

lemma *inv_max_hi_delete*:
  inv_max_hi t ⟹ *inv_max_hi* (*delete x t*)
apply (*induct t*)
 apply *simp*
using *inv_max_hi_split_min* **by** (*fastforce simp add*: *Let_def split*: *prod.splits*)

14.4 Search

Does interval x overlap with any interval in the tree?

fun *search* :: *'a*::{*linorder*,*order_bot*} *ivl_tree* ⇒ *'a ivl* ⇒ *bool* **where**
search Leaf x = *False* |

```
search (Node l (a, m) r) x =
  (if overlap x a then True
   else if l ≠ Leaf ∧ max_hi l ≥ low x then search l x
   else search r x)

lemma search_correct:
  inv_max_hi t ⟹ sorted (inorder t) ⟹ search t x = has_overlap (set_tree t) x
proof (induction t rule: tree2_induct)
  case Leaf
  then show ?case by (auto simp add: has_overlap_def)
next
  case (Node l a m r)
  have search_l: search l x = has_overlap (set_tree l) x
    using Node.IH(1) Node.prems by (auto simp: sorted_wrt_append)
  have search_r: search r x = has_overlap (set_tree r) x
    using Node.IH(2) Node.prems by (auto simp: sorted_wrt_append)
  show ?case
  proof (cases overlap a x)
    case True
    thus ?thesis by (auto simp: overlap_def has_overlap_def)
  next
    case a_disjoint: False
    then show ?thesis
    proof cases
      assume [simp]: l = Leaf
      have search_eval: search (Node l (a, m) r) x = search r x
        using a_disjoint overlap_def by auto
      show ?thesis
        unfolding search_eval search_r
        by (auto simp add: has_overlap_def a_disjoint)
    next
      assume l ≠ Leaf
      then show ?thesis
      proof (cases max_hi l ≥ low x)
        case max_hi_l_ge: True
        have inv_max_hi l
          using Node.prems(1) by auto
        then obtain p where p: p ∈ set_tree l high p = max_hi l
          using ‹l ≠ Leaf› max_hi_exists by auto
        have search_eval: search (Node l (a, m) r) x = search l x
           using a_disjoint ‹l ≠ Leaf› max_hi_l_ge by (auto simp: overlap_def)
        show ?thesis
```

```
        proof (cases low p ≤ high x)
          case True
          have overlap p x
            unfolding overlap_def using True p(2) max_hi_l_ge by auto
          then show ?thesis
            unfolding search_eval search_l
            using p(1) by(auto simp: has_overlap_def overlap_def)
        next
          case False
          have ¬overlap x rp if asm: rp ∈ set_tree r for rp
          proof −
            have low p ≤ low rp
             using asm p(1) Node(4) by(fastforce simp: sorted_wrt_append
ivl_less)
            then show ?thesis
              using False by (auto simp: overlap_def)
          qed
          then show ?thesis
            unfolding search_eval search_l
            using a_disjoint by (auto simp: has_overlap_def overlap_def)
        qed
      next
        case False
        have search_eval: search (Node l (a, m) r) x = search r x
          using a_disjoint False by (auto simp: overlap_def)
        have ¬overlap x lp if asm: lp ∈ set_tree l for lp
          using asm False Node.prems(1) max_hi_is_max
          by (fastforce simp: overlap_def)
        then show ?thesis
          unfolding search_eval search_r
          using a_disjoint by (auto simp: has_overlap_def overlap_def)
      qed
    qed
  qed
qed

definition empty :: 'a ivl_tree where
empty = Leaf
```

14.5 Specification

```
locale Interval_Set = Set +
  fixes has_overlap :: 't ⇒ 'a::linorder ivl ⇒ bool
 assumes set_overlap: invar s ⟹ has_overlap s x = Interval_Tree.has_overlap
```

(*set s*) *x*

fun *invar* :: ($'a$::{*linorder*,*order_bot*}) *ivl_tree* ⇒ *bool* **where**
invar t = (*inv_max_hi t* ∧ *sorted*(*inorder t*))

interpretation *S*: *Interval_Set*
  where *empty* = *Leaf* **and** *insert* = *insert* **and** *delete* = *delete*
  and *has_overlap* = *search* **and** *isin* = *isin* **and** *set* = *set_tree*
  and *invar* = *invar*
proof (*standard*, *goal_cases*)
  case *1*
  then show *?case* **by** *auto*
next
  case *2*
  then show *?case* **by** (*simp add*: *isin_set_inorder*)
next
  case *3*
  then show *?case* **by**(*simp add*: *inorder_insert set_ins_list flip*: *set_inorder*)
next
  case *4*
  then show *?case* **by**(*simp add*: *inorder_delete set_del_list flip*: *set_inorder*)
next
  case *5*
  then show *?case* **by** *auto*
next
  case *6*
  then show *?case* **by** (*simp add*: *inorder_insert inv_max_hi_insert sorted_ins_list*)
next
  case *7*
  then show *?case* **by** (*simp add*: *inorder_delete inv_max_hi_delete sorted_del_list*)
next
  case *8*
  then show *?case* **by** (*simp add*: *search_correct*)
qed

end

15 AVL Tree Implementation of Sets

theory *AVL_Set_Code*
imports
  Cmp
  Isin2

begin

15.1 Code

```
type_synonym 'a tree_ht = ('a*nat) tree

definition empty :: 'a tree_ht where
empty = Leaf

fun ht :: 'a tree_ht ⇒ nat where
ht Leaf = 0 |
ht (Node l (a,n) r) = n

definition node :: 'a tree_ht ⇒ 'a ⇒ 'a tree_ht ⇒ 'a tree_ht where
node l a r = Node l (a, max (ht l) (ht r) + 1) r

definition balL :: 'a tree_ht ⇒ 'a ⇒ 'a tree_ht ⇒ 'a tree_ht where
balL AB c C =
  (if ht AB = ht C + 2 then
     case AB of
       Node A (a, _) B ⇒
         if ht A ≥ ht B then node A a (node B c C)
         else
           case B of
             Node B₁ (b, _) B₂ ⇒ node (node A a B₁) b (node B₂ c C)
   else node AB c C)

definition balR :: 'a tree_ht ⇒ 'a ⇒ 'a tree_ht ⇒ 'a tree_ht where
balR A a BC =
   (if ht BC = ht A + 2 then
      case BC of
        Node B (c, _) C ⇒
          if ht B ≤ ht C then node (node A a B) c C
          else
            case B of
              Node B₁ (b, _) B₂ ⇒ node (node A a B₁) b (node B₂ c C)
  else node A a BC)

fun insert :: 'a::linorder ⇒ 'a tree_ht ⇒ 'a tree_ht where
insert x Leaf = Node Leaf (x, 1) Leaf |
insert x (Node l (a, n) r) = (case cmp x a of
   EQ ⇒ Node l (a, n) r |
   LT ⇒ balL (insert x l) a r |
   GT ⇒ balR l a (insert x r))
```

```
fun split_max :: 'a tree_ht ⇒ 'a tree_ht * 'a where
split_max (Node l (a, _) r) =
  (if r = Leaf then (l,a) else let (r',a') = split_max r in (balL l a r', a'))

lemmas split_max_induct = split_max.induct[case_names Node Leaf]

fun delete :: 'a::linorder ⇒ 'a tree_ht ⇒ 'a tree_ht where
delete _ Leaf = Leaf |
delete x (Node l (a, n) r) =
  (case cmp x a of
     EQ ⇒ if l = Leaf then r
           else let (l', a') = split_max l in balR l' a' r |
     LT ⇒ balR (delete x l) a r |
     GT ⇒ balL l a (delete x r))
```

15.2 Functional Correctness Proofs

Very different from the AFP/AVL proofs

15.2.1 Proofs for insert

```
lemma inorder_balL:
  inorder (balL l a r) = inorder l @ a # inorder r
by (auto simp: node_def balL_def split:tree.splits)

lemma inorder_balR:
  inorder (balR l a r) = inorder l @ a # inorder r
by (auto simp: node_def balR_def split:tree.splits)

theorem inorder_insert:
  sorted(inorder t) ⟹ inorder(insert x t) = ins_list x (inorder t)
by (induct t)
   (auto simp: ins_list_simps inorder_balL inorder_balR)
```

15.2.2 Proofs for delete

```
lemma inorder_split_maxD:
  ⟦ split_max t = (t',a); t ≠ Leaf ⟧ ⟹
   inorder t' @ [a] = inorder t
by(induction t arbitrary: t' rule: split_max.induct)
  (auto simp: inorder_balL split: if_splits prod.splits tree.split)

theorem inorder_delete:
```

sorted(*inorder t*) $\Longrightarrow$ *inorder* (*delete x t*) = *del_list x* (*inorder t*)
by(*induction t*)
(*auto simp*: *del_list_simps inorder_balL inorder_balR inorder_split_maxD split*: *prod.splits*)

end

15.3 Invariant

theory *AVL_Set*
imports
AVL_Set_Code
HOL−Number_Theory.Fib
begin

fun *avl* :: *'a tree_ht* ⇒ *bool* **where**
avl Leaf = *True* |
avl (*Node l* (*a*,*n*) *r*) =
(*abs*(*int*(*height l*) − *int*(*height r*)) ≤ *1* ∧
n = *max* (*height l*) (*height r*) + *1* ∧ *avl l* ∧ *avl r*)

15.3.1 Insertion maintains AVL balance

declare *Let_def* [*simp*]

lemma *ht_height*[*simp*]: *avl t* $\Longrightarrow$ *ht t* = *height t*
by (*cases t rule*: *tree2_cases*) *simp_all*

First, a fast but relatively manual proof with many lemmas:

lemma *height_balL*:
⟦ *avl l*; *avl r*; *height l* = *height r* + *2* ⟧ $\Longrightarrow$
height (*balL l a r*) ∈ {*height r* + *2*, *height r* + *3*}
by (*auto simp*:*node_def balL_def split*:*tree.split*)

lemma *height_balR*:
⟦ *avl l*; *avl r*; *height r* = *height l* + *2* ⟧ $\Longrightarrow$
height (*balR l a r*) : {*height l* + *2*, *height l* + *3*}
by(*auto simp add*:*node_def balR_def split*:*tree.split*)

lemma *height_node*[*simp*]: *height*(*node l a r*) = *max* (*height l*) (*height r*) + *1*
by (*simp add*: *node_def*)

lemma *height_balL2*:
⟦ *avl l*; *avl r*; *height l* ≠ *height r* + *2* ⟧ $\Longrightarrow$

height (*balL l a r*) = *1* + *max* (*height l*) (*height r*)
by (*simp_all add*: *balL_def*)

lemma *height_balR2*:
⟦ *avl l*; *avl r*; *height r* ≠ *height l* + *2* ⟧ ⟹
height (*balR l a r*) = *1* + *max* (*height l*) (*height r*)
by (*simp_all add*: *balR_def*)

lemma *avl_balL*:
⟦ *avl l*; *avl r*; *height r* − *1* ≤ *height l* ∧ *height l* ≤ *height r* + *2* ⟧ ⟹
avl(*balL l a r*)
by(*auto simp*: *balL_def node_def split*!: *if_split tree.split*)

lemma *avl_balR*:
⟦ *avl l*; *avl r*; *height l* − *1* ≤ *height r* ∧ *height r* ≤ *height l* + *2* ⟧ ⟹
avl(*balR l a r*)
by(*auto simp*: *balR_def node_def split*!: *if_split tree.split*)

Insertion maintains the AVL property. Requires simultaneous proof.

theorem *avl_insert*:
avl t ⟹ *avl*(*insert x t*)
avl t ⟹ *height* (*insert x t*) ∈ {*height t*, *height t* + *1*}
proof (*induction t rule*: *tree2_induct*)
case (*Node l a _ r*)
case *1*
show *?case*
proof(*cases x* = *a*)
case *True* **with** *1* **show** *?thesis* **by** *simp*
next
case *False*
show *?thesis*
proof(*cases x*<*a*)
case *True* **with** *1 Node*(*1*,*2*) **show** *?thesis* **by** (*auto intro*!:*avl_balL*)
next
case *False* **with** *1 Node*(*3*,*4*) ‹*x*≠*a*› **show** *?thesis* **by** (*auto intro*!:*avl_balR*)
qed
qed
case *2*
show *?case*
proof(*cases x* = *a*)
case *True* **with** *2* **show** *?thesis* **by** *simp*
next
case *False*

```
    show ?thesis
    proof(cases x<a)
      case True
      show ?thesis
      proof(cases height (insert x l) = height r + 2)
        case False with 2 Node(1,2) ‹x < a› show ?thesis by (auto simp:
height_balL2)
      next
        case True
        hence (height (balL (insert x l) a r) = height r + 2) ∨
          (height (balL (insert x l) a r) = height r + 3) (is ?A ∨ ?B)
          using 2 Node(1,2) height_balL[OF _ _ True] by simp
        thus ?thesis
        proof
          assume ?A with 2 ‹x < a› show ?thesis by (auto)
        next
          assume ?B with 2 Node(2) True ‹x < a› show ?thesis by (simp)
arith
        qed
      qed
    next
      case False
      show ?thesis
      proof(cases height (insert x r) = height l + 2)
        case False with 2 Node(3,4) ‹¬x < a› show ?thesis by (auto simp:
height_balR2)
      next
        case True
        hence (height (balR l a (insert x r)) = height l + 2) ∨
          (height (balR l a (insert x r)) = height l + 3)  (is ?A ∨ ?B)
          using 2 Node(3) height_balR[OF _ _ True] by simp
        thus ?thesis
        proof
          assume ?A with 2 ‹¬x < a› show ?thesis by (auto)
        next
          assume ?B with 2 Node(4) True ‹¬x < a› show ?thesis by (simp)
arith
        qed
      qed
    qed
  qed
qed simp_all
```

Now an automatic proof without lemmas:

```
theorem avl_insert_auto: avl t ⟹
  avl(insert x t) ∧ height (insert x t) ∈ {height t, height t + 1}
apply (induction t rule: tree2_induct)

 apply (auto simp: balL_def balR_def node_def max_absorb2 split!: if_split
tree.split)
done
```

15.3.2 Deletion maintains AVL balance

```
lemma avl_split_max:
  ⟦ avl t; t ≠ Leaf ⟧ ⟹
  avl (fst (split_max t)) ∧
  height t ∈ {height(fst (split_max t)), height(fst (split_max t)) + 1}
by(induct t rule: split_max_induct)
   (auto simp: balL_def node_def max_absorb2 split!: prod.split if_split
tree.split)
```

Deletion maintains the AVL property:

```
theorem avl_delete:
  avl t ⟹ avl(delete x t)
  avl t ⟹ height t ∈ {height (delete x t), height (delete x t) + 1}
proof (induct t rule: tree2_induct)
  case (Node l a n r)
  case 1
  show ?case
  proof(cases x = a)
    case True thus ?thesis
      using 1 avl_split_max[of l] by (auto intro!: avl_balR split: prod.split)
  next
    case False thus ?thesis
      using Node 1 by (auto intro!: avl_balL avl_balR)
  qed
  case 2
  show ?case
  proof(cases x = a)
    case True thus ?thesis using 2 avl_split_max[of l]
    by(auto simp: balR_def max_absorb2 split!: if_splits prod.split tree.split)
  next
    case False
    show ?thesis
    proof(cases x<a)
      case True
      show ?thesis
```

proof(*cases height r* = *height* (*delete x l*) + *2*)
case *False*
thus *?thesis* **using** *2 Node*(*1*,*2*) ‹*x* < *a*› **by**(*auto simp*: *balR_def*)
next
case *True*
thus *?thesis* **using** *height_balR*[*OF* _ _ *True*, *of a*] *2 Node*(*1*,*2*) ‹*x* < *a*› **by** *simp linarith*
qed
next
case *False*
show *?thesis*
proof(*cases height l* = *height* (*delete x r*) + *2*)
case *False*
thus *?thesis* **using** *2 Node*(*3*,*4*) ‹¬*x* < *a*› ‹*x* ≠ *a*› **by**(*auto simp*: *balL_def*)
next
case *True*
thus *?thesis*
using *height_balL*[*OF* _ _ *True*, *of a*] *2 Node*(*3*,*4*) ‹¬*x* < *a*› ‹*x* ≠ *a*› **by** *simp linarith*
qed
qed
qed
qed *simp_all*

A more automatic proof. Complete automation as for insertion seems hard due to resource requirements.

theorem *avl_delete_auto*:
avl t ⟹ *avl*(*delete x t*)
avl t ⟹ *height t* ∈ {*height* (*delete x t*), *height* (*delete x t*) + *1*}
proof (*induct t rule*: *tree2_induct*)
case (*Node l a n r*)
case *1*
thus *?case*
using *Node avl_split_max*[*of l*] **by** (*auto intro*!: *avl_balL avl_balR split*: *prod.split*)
case *2*
show *?case*
using *2 Node avl_split_max*[*of l*]
by *auto*
(*auto simp*: *balL_def balR_def max_absorb1 max_absorb2 split*!: *tree.splits prod.splits if_splits*)
qed *simp_all*

15.4 Overall correctness

interpretation *S*: *Set_by_Ordered*
where *empty* = *empty* **and** *isin* = *isin* **and** *insert* = *insert* **and** *delete* = *delete*
and *inorder* = *inorder* **and** *inv* = *avl*
proof (*standard*, *goal_cases*)
  case *1* **show** *?case* **by** (*simp add*: *empty_def*)
next
  case *2* **thus** *?case* **by**(*simp add*: *isin_set_inorder*)
next
  case *3* **thus** *?case* **by**(*simp add*: *inorder_insert*)
next
  case *4* **thus** *?case* **by**(*simp add*: *inorder_delete*)
next
  case *5* **thus** *?case* **by** (*simp add*: *empty_def*)
next
  case *6* **thus** *?case* **by** (*simp add*: *avl_insert*(*1*))
next
  case *7* **thus** *?case* **by** (*simp add*: *avl_delete*(*1*))
qed

15.5 Height-Size Relation

Any AVL tree of height n has at least *fib* $(n+2)$ leaves:

theorem *avl_fib_bound*:
  avl t $\Longrightarrow$ *fib*(*height t* + *2*) $\leq$ *size1 t*
proof (*induction rule*: *tree2_induct*)
  case (*Node l a h r*)
  have *1*: *height l* + *1* $\leq$ *height r* + *2* **and** *2*: *height r* + *1* $\leq$ *height l* + *2*
    using *Node.prems* **by** *auto*
  have *fib* (*max* (*height l*) (*height r*) + *3*) $\leq$ *size1 l* + *size1 r*
  proof *cases*
    assume *height l* $\geq$ *height r*
    hence *fib* (*max* (*height l*) (*height r*) + *3*) = *fib* (*height l* + *3*)
      by(*simp add*: *max_absorb1*)
    also have ... = *fib* (*height l* + *2*) + *fib* (*height l* + *1*)
      by (*simp add*: *numeral_eq_Suc*)
    also have ... $\leq$ *size1 l* + *fib* (*height l* + *1*)
      using *Node* **by** (*simp*)
    also have ... $\leq$ *size1 r* + *size1 l*
      using *Node fib_mono*[*OF 1*] **by** *auto*
    also have ... = *size1* (*Node l* (*a*,*h*) *r*)
      by *simp*

```
    finally show ?thesis
      by (simp)
  next
    assume ¬ height l ≥ height r
    hence fib (max (height l) (height r) + 3) = fib (height r + 3)
      by(simp add: max_absorb1)
    also have ... = fib (height r + 2) + fib (height r + 1)
      by (simp add: numeral_eq_Suc)
    also have ... ≤ size1 r + fib (height r + 1)
      using Node by (simp)
    also have ... ≤ size1 r + size1 l
      using Node fib_mono[OF 2] by auto
    also have ... = size1 (Node l (a,h) r)
      by simp
    finally show ?thesis
      by (simp)
  qed
  also have ... = size1 (Node l (a,h) r)
    by simp
  finally show ?case by (simp del: fib.simps add: numeral_eq_Suc)
qed auto

lemma avl_fib_bound_auto: avl t ⟹ fib (height t + 2) ≤ size1 t
proof (induction t rule: tree2_induct)
  case Leaf thus ?case by (simp)
next
  case (Node l a h r)
  have 1: height l + 1 ≤ height r + 2 and 2: height r + 1 ≤ height l + 2
    using Node.prems by auto
  have left: height l ≥ height r ⟹ ?case (is ?asm ⟹ _)
    using Node fib_mono[OF 1] by (simp add: max.absorb1)
  have right: height l ≤ height r ⟹ ?case
    using Node fib_mono[OF 2] by (simp add: max.absorb2)
  show ?case using left right using Node.prems by simp linarith
qed
```

An exponential lower bound for *fib*:

```
lemma fib_lowerbound:
  defines φ ≡ (1 + sqrt 5) / 2
  shows real (fib(n+2)) ≥ φ ^ n
proof (induction n rule: fib.induct)
  case 1
  then show ?case by simp
next
```

```
  case 2
  then show ?case by (simp add: φ_def real_le_lsqrt)
next
  case (3 n)
  have φ ^ Suc (Suc n) = φ ^ 2 * φ ^ n
    by (simp add: field_simps power2_eq_square)
  also have ... = (φ + 1) * φ ^ n
    by (simp_all add: φ_def power2_eq_square field_simps)
  also have ... = φ ^ Suc n + φ ^ n
      by (simp add: field_simps)
  also have ... ≤ real (fib (Suc n + 2)) + real (fib (n + 2))
      by (intro add_mono 3.IH)
  finally show ?case by simp
qed
```

The size of an AVL tree is (at least) exponential in its height:

```
lemma avl_size_lowerbound:
  defines φ ≡ (1 + sqrt 5) / 2
  assumes avl t
  shows   φ ^(height t) ≤ size1 t
proof –
  have φ ^ height t ≤ fib (height t + 2)
    unfolding φ_def by(rule fib_lowerbound)
  also have ... ≤ size1 t
    using avl_fib_bound[of t] assms by simp
  finally show ?thesis .
qed
```

The height of an AVL tree is most $1 / log\ 2\ \varphi \approx 1.44$ times worse than $log\ 2\ (real\ (size1\ t))$:

```
lemma  avl_height_upperbound:
  defines φ ≡ (1 + sqrt 5) / 2
  assumes avl t
  shows   height t ≤ (1/log 2 φ) * log 2 (size1 t)
proof –
  have φ > 0 φ > 1 by(auto simp: φ_def pos_add_strict)
  hence height t = log φ (φ ^ height t) by(simp add: log_nat_power)
  also have ... ≤ log φ (size1 t)
    using avl_size_lowerbound[OF assms(2), folded φ_def] ‹1 < φ›
    by (simp add: le_log_of_power)
  also have ... = (1/log 2 φ) * log 2 (size1 t)
    by(simp add: log_base_change[of 2 φ])
  finally show ?thesis .
qed
```

end

16 Function *lookup* for Tree2

theory *Lookup2*
imports
Tree2
Cmp
Map_Specs
begin

fun *lookup* :: ((′*a*::*linorder* ∗ ′*b*) ∗ ′*c*) *tree* ⇒ ′*a* ⇒ ′*b option* **where**
lookup Leaf x = *None* |
lookup (*Node l* ((*a*,*b*), _) *r*) *x* =
(*case cmp x a of LT* ⇒ *lookup l x* | *GT* ⇒ *lookup r x* | *EQ* ⇒ *Some b*)

lemma *lookup_map_of*:
sorted1(*inorder t*) ⟹ *lookup t x* = *map_of* (*inorder t*) *x*
by(*induction t rule*: *tree2_induct*) (*auto simp*: *map_of_simps split*: *option.split*)

end

17 AVL Tree Implementation of Maps

theory *AVL_Map*
imports
AVL_Set
Lookup2
begin

fun *update* :: ′*a*::*linorder* ⇒ ′*b* ⇒ (′*a*∗′*b*) *tree_ht* ⇒ (′*a*∗′*b*) *tree_ht* **where**
update x y Leaf = *Node Leaf* ((*x*,*y*), *1*) *Leaf* |
update x y (*Node l* ((*a*,*b*), *h*) *r*) = (*case cmp x a of*
EQ ⇒ *Node l* ((*x*,*y*), *h*) *r* |
LT ⇒ *balL* (*update x y l*) (*a*,*b*) *r* |
GT ⇒ *balR l* (*a*,*b*) (*update x y r*))

fun *delete* :: ′*a*::*linorder* ⇒ (′*a*∗′*b*) *tree_ht* ⇒ (′*a*∗′*b*) *tree_ht* **where**
delete _ *Leaf* = *Leaf* |
delete x (*Node l* ((*a*,*b*), *h*) *r*) = (*case cmp x a of*
EQ ⇒ *if l* = *Leaf then r*

```
      else let (l', ab') = split_max l in balR l' ab' r |
   LT ⇒ balR (delete x l) (a,b) r |
   GT ⇒ balL l (a,b) (delete x r))
```

17.1 Functional Correctness

```
theorem inorder_update:
  sorted1(inorder t) ⟹ inorder(update x y t) = upd_list x y (inorder t)
by (induct t) (auto simp: upd_list_simps inorder_balL inorder_balR)


theorem inorder_delete:
  sorted1(inorder t) ⟹ inorder (delete x t) = del_list x (inorder t)
by(induction t)
  (auto simp: del_list_simps inorder_balL inorder_balR
     inorder_split_maxD split: prod.splits)
```

17.2 AVL invariants

17.2.1 Insertion maintains AVL balance

```
theorem avl_update:
  assumes avl t
  shows avl(update x y t)
       (height (update x y t) = height t ∨ height (update x y t) = height t
+ 1)
using assms
proof (induction x y t rule: update.induct)
  case eq2: (2 x y l a b h r)
  case 1
  show ?case
  proof(cases x = a)
    case True with eq2 1 show ?thesis by simp
  next
    case False
    with eq2 1 show ?thesis
    proof(cases x<a)
      case True with eq2 1 show ?thesis by (auto intro!: avl_balL)
    next
      case False with eq2 1 ‹x≠a› show ?thesis by (auto intro!: avl_balR)
    qed
  qed
  case 2
  show ?case
  proof(cases x = a)
```

```
    case True with eq2 1 show ?thesis by simp
  next
    case False
    show ?thesis
    proof(cases x<a)
      case True
      show ?thesis
      proof(cases height (update x y l) = height r + 2)
        case False with eq2 2 ‹x < a› show ?thesis by (auto simp: height_balL2)
      next
        case True
        hence (height (balL (update x y l) (a,b) r) = height r + 2) ∨
          (height (balL (update x y l) (a,b) r) = height r + 3) (is ?A ∨ ?B)
          using eq2 2 ‹x<a› height_balL[OF _ _ True] by simp
        thus ?thesis
        proof
          assume ?A with 2 ‹x < a› show ?thesis by (auto)
        next
          assume ?B with True 1 eq2(2) ‹x < a› show ?thesis by (simp) arith
        qed
      qed
    next
      case False
      show ?thesis
      proof(cases height (update x y r) = height l + 2)
        case False with eq2 2 ‹¬x < a› show ?thesis by (auto simp: height_balR2)
      next
        case True
        hence (height (balR l (a,b) (update x y r)) = height l + 2) ∨
          (height (balR l (a,b) (update x y r)) = height l + 3)  (is ?A ∨ ?B)
          using eq2 2 ‹¬x < a› ‹x ≠ a› height_balR[OF _ _ True] by simp
        thus ?thesis
        proof
          assume ?A with 2 ‹¬x < a› show ?thesis by (auto)
        next
          assume ?B with True 1 eq2(4) ‹¬x < a› show ?thesis by (simp) arith
        qed
      qed
    qed
  qed
```

```
qed simp_all
```

17.2.2 Deletion maintains AVL balance

```
theorem avl_delete:
  assumes avl t
  shows avl(delete x t) and height t = (height (delete x t)) ∨ height t =
height (delete x t) + 1
using assms
proof (induct t rule: tree2_induct)
  case (Node l ab h r)
  obtain a b where [simp]: ab = (a,b) by fastforce
  case 1
  show ?case
  proof(cases x = a)
    case True with Node 1 show ?thesis
      using avl_split_max[of l] by (auto intro!: avl_balR split: prod.split)
  next
    case False
    show ?thesis
    proof(cases x<a)
      case True with Node 1 show ?thesis by (auto intro!: avl_balR)
    next
     case False with Node 1 ‹x≠a› show ?thesis by (auto intro!: avl_balL)
    qed
  qed
  case 2
  show ?case
  proof(cases x = a)
    case True then show ?thesis using 1 avl_split_max[of l]
    by(auto simp: balR_def max_absorb2 split!: if_splits prod.split tree.split)
  next
    case False
    show ?thesis
    proof(cases x<a)
      case True
      show ?thesis
      proof(cases height r = height (delete x l) + 2)
          case False with Node 1 ‹x < a› show ?thesis by(auto simp:
balR_def)
      next
        case True
       thus ?thesis using height_balR[OF _ _ True, of ab] 2 Node(1,2) ‹x
< a› by simp linarith
```

qed
next
case *False*
show *?thesis*
proof(*cases height l* = *height* (*delete x r*) + *2*)
case *False* **with** *Node 1* ‹¬*x* < *a*› ‹*x* ≠ *a*› **show** *?thesis* **by**(*auto simp*: *balL_def*)
next
case *True*
thus *?thesis*
using *height_balL*[*OF _ _ True*, *of ab*] *2* *Node*(*3*,*4*) ‹¬*x* < *a*› ‹*x* ≠ *a*› **by** *auto*
qed
qed
qed
qed *simp_all*

interpretation *M*: *Map_by_Ordered*
where *empty* = *empty* **and** *lookup* = *lookup* **and** *update* = *update* **and** *delete* = *delete*
and *inorder* = *inorder* **and** *inv* = *avl*
proof (*standard*, *goal_cases*)
case *1* **show** *?case* **by** (*simp add*: *empty_def*)
next
case *2* **thus** *?case* **by**(*simp add*: *lookup_map_of*)
next
case *3* **thus** *?case* **by**(*simp add*: *inorder_update*)
next
case *4* **thus** *?case* **by**(*simp add*: *inorder_delete*)
next
case *5* **show** *?case* **by** (*simp add*: *empty_def*)
next
case *6* **thus** *?case* **by**(*simp add*: *avl_update*(*1*))
next
case *7* **thus** *?case* **by**(*simp add*: *avl_delete*(*1*))
qed

end

18 AVL Tree with Balance Factors (1)

theory *AVL_Bal_Set*

imports
  Cmp
  Isin2
begin

This version detects height increase/decrease from above via the change in balance factors.

datatype *bal* = *Lh* | *Bal* | *Rh*

type_synonym *'a tree_bal* = (*'a* ∗ *bal*) *tree*

Invariant:

```
fun avl :: 'a tree_bal ⇒ bool where
avl Leaf = True |
avl (Node l (a,b) r) =
  ((case b of
    Bal ⇒ height r = height l |
    Lh ⇒ height l = height r + 1 |
    Rh ⇒ height r = height l + 1)
  ∧ avl l ∧ avl r)
```

18.1 Code

```
fun is_bal where
is_bal (Node l (a,b) r) = (b = Bal)

fun incr where
incr t t' = (t = Leaf ∨ is_bal t ∧ ¬ is_bal t')

fun rot2 where
rot2 A a B c C = (case B of
  (Node B₁ (b, bb) B₂) ⇒
    let b₁ = if bb = Rh then Lh else Bal;
        b₂ = if bb = Lh then Rh else Bal
    in Node (Node A (a,b₁) B₁) (b,Bal) (Node B₂ (c,b₂) C))

fun balL :: 'a tree_bal ⇒ 'a ⇒ bal ⇒ 'a tree_bal ⇒ 'a tree_bal where
balL AB c bc C = (case bc of
     Bal ⇒ Node AB (c,Lh) C |
     Rh ⇒ Node AB (c,Bal) C |
     Lh ⇒ (case AB of
       Node A (a,Lh) B ⇒ Node A (a,Bal) (Node B (c,Bal) C) |
       Node A (a,Bal) B ⇒ Node A (a,Rh) (Node B (c,Lh) C) |
       Node A (a,Rh) B ⇒ rot2 A a B c C))
```

fun *balR* :: *'a tree_bal* ⇒ *'a* ⇒ *bal* ⇒ *'a tree_bal* ⇒ *'a tree_bal* **where**
balR A a ba BC = (*case ba of*
  Bal ⇒ *Node A* (*a*,*Rh*) *BC* |
  Lh ⇒ *Node A* (*a*,*Bal*) *BC* |
  Rh ⇒ (*case BC of*
    Node B (*c*,*Rh*) *C* ⇒ *Node* (*Node A* (*a*,*Bal*) *B*) (*c*,*Bal*) *C* |
    Node B (*c*,*Bal*) *C* ⇒ *Node* (*Node A* (*a*,*Rh*) *B*) (*c*,*Lh*) *C* |
    Node B (*c*,*Lh*) *C* ⇒ *rot2 A a B c C*))

fun *insert* :: *'a::linorder* ⇒ *'a tree_bal* ⇒ *'a tree_bal* **where**
insert x Leaf = *Node Leaf* (*x*, *Bal*) *Leaf* |
insert x (*Node l* (*a*, *b*) *r*) = (*case cmp x a of*
  EQ ⇒ *Node l* (*a*, *b*) *r* |
  LT ⇒ *let l'* = *insert x l in if incr l l' then balL l' a b r else Node l'* (*a*,*b*) *r* |
  GT ⇒ *let r'* = *insert x r in if incr r r' then balR l a b r' else Node l* (*a*,*b*) *r'*)

fun *decr* **where**
decr t t' = (*t* ≠ *Leaf* ∧ (*t'* = *Leaf* ∨ ¬ *is_bal t* ∧ *is_bal t'*))

fun *split_max* :: *'a tree_bal* ⇒ *'a tree_bal* ∗ *'a* **where**
split_max (*Node l* (*a*, *ba*) *r*) =
 (*if r* = *Leaf then* (*l*,*a*)
  else let (*r'*,*a'*) = *split_max r*;
      t' = *if decr r r' then balL l a ba r' else Node l* (*a*,*ba*) *r'*
    in (*t'*, *a'*))

fun *delete* :: *'a::linorder* ⇒ *'a tree_bal* ⇒ *'a tree_bal* **where**
delete _ Leaf = *Leaf* |
delete x (*Node l* (*a*, *ba*) *r*) =
 (*case cmp x a of*
  EQ ⇒ *if l* = *Leaf then r*
    else let (*l'*, *a'*) = *split_max l in*
      if decr l l' then balR l' a' ba r else Node l' (*a'*,*ba*) *r* |
  LT ⇒ *let l'* = *delete x l in if decr l l' then balR l' a ba r else Node l'* (*a*,*ba*) *r* |
  GT ⇒ *let r'* = *delete x r in if decr r r' then balL l a ba r' else Node l* (*a*,*ba*) *r'*)

18.2 Proofs

lemmas *split_max_induct* = *split_max.induct*[*case_names Node Leaf*]

lemmas *splits* = *if_splits tree.splits bal.splits*

declare *Let_def* [*simp*]

18.2.1 Proofs about insertion

lemma *avl_insert*: *avl t* ⟹
avl(*insert x t*) ∧
height(*insert x t*) = *height t* + (*if incr t* (*insert x t*) *then 1 else 0*)
apply(*induction x t rule*: *insert.induct*)
apply(*auto split*!: *splits*)
done

The following two auxiliary lemma merely simplify the proof of *inorder_insert*.

lemma [*simp*]: [] ≠ *ins_list x xs*
by(*cases xs*) *auto*

lemma [*simp*]: *avl t* ⟹ *insert x t* ≠ ⟨*l*, (*a*, *Rh*), ⟨⟩⟩ ∧ *insert x t* ≠ ⟨⟨⟩, (*a*, *Lh*), *r*⟩
by(*drule avl_insert*[*of _ x*]) (*auto split*: *splits*)

theorem *inorder_insert*:
⟦ *avl t*; *sorted*(*inorder t*) ⟧ ⟹ *inorder*(*insert x t*) = *ins_list x* (*inorder t*)
apply(*induction t*)
apply (*auto simp*: *ins_list_simps split*!: *splits*)
done

18.2.2 Proofs about deletion

lemma *inorder_balR*:
⟦ *ba* = *Rh* ⟶ *r* ≠ *Leaf*; *avl r* ⟧
⟹ *inorder* (*balR l a ba r*) = *inorder l* @ *a* # *inorder r*
by (*auto split*: *splits*)

lemma *inorder_balL*:
⟦ *ba* = *Lh* ⟶ *l* ≠ *Leaf*; *avl l* ⟧
⟹ *inorder* (*balL l a ba r*) = *inorder l* @ *a* # *inorder r*
by (*auto split*: *splits*)

lemma *height_1_iff*: *avl t* ⟹ *height t* = *Suc 0* ⟷ (∃ *x*. *t* = *Node Leaf* (*x*,*Bal*) *Leaf*)
by(*cases t*) (*auto split*: *splits prod.splits*)

lemma *avl_split_max*:
⟦ *split_max t* = (*t'*,*a*); *avl t*; *t* ≠ *Leaf* ⟧ ⟹
avl t' ∧ *height t* = *height t'* + (*if decr t t' then 1 else 0*)
apply(*induction t arbitrary*: *t' a rule*: *split_max_induct*)
apply(*auto simp*: *max_absorb1 max_absorb2 height_1_iff split*!: *splits prod.splits*)
done

lemma *avl_delete*: *avl t* ⟹
avl (*delete x t*) ∧
height t = *height* (*delete x t*) + (*if decr t* (*delete x t*) *then 1 else 0*)
apply(*induction x t rule*: *delete.induct*)
apply(*auto simp*: *max_absorb1 max_absorb2 height_1_iff dest*: *avl_split_max split*!: *splits prod.splits*)
done

lemma *inorder_split_maxD*:
⟦ *split_max t* = (*t'*,*a*); *t* ≠ *Leaf*; *avl t* ⟧ ⟹
inorder t' @ [*a*] = *inorder t*
apply(*induction t arbitrary*: *t' rule*: *split_max.induct*)
apply(*fastforce split*!: *splits prod.splits*)
apply *simp*
done

lemma *neq_Leaf_if_height_neq_0*: *height t* ≠ *0* ⟹ *t* ≠ *Leaf*
by *auto*

lemma *split_max_Leaf*: ⟦ *t* ≠ *Leaf*; *avl t* ⟧ ⟹ *split_max t* = (⟨⟩, *x*) ⟷ *t* = *Node Leaf* (*x*,*Bal*) *Leaf*
by(*cases t*) (*auto split*: *splits prod.splits*)

theorem *inorder_delete*:
⟦ *avl t*; *sorted*(*inorder t*) ⟧ ⟹ *inorder* (*delete x t*) = *del_list x* (*inorder t*)
apply(*induction t rule*: *tree2_induct*)
apply(*auto simp*: *del_list_simps inorder_balR inorder_balL avl_delete inorder_split_maxD*
split_max_Leaf neq_Leaf_if_height_neq_0
simp del: *balL.simps balR.simps split*!: *splits prod.splits*)
done

18.2.3 Set Implementation

interpretation *S*: *Set_by_Ordered*
where *empty* = *Leaf* **and** *isin* = *isin*
  and *insert* = *insert*
  and *delete* = *delete*
  and *inorder* = *inorder* **and** *inv* = *avl*
proof (*standard*, *goal_cases*)
  case *1* **show** *?case* **by** (*simp*)
next
  case *2* **thus** *?case* **by**(*simp add*: *isin_set_inorder*)
next
  case *3* **thus** *?case* **by**(*simp add*: *inorder_insert*)
next
  case *4* **thus** *?case* **by**(*simp add*: *inorder_delete*)
next
  case *5* **thus** *?case* **by** (*simp*)
next
  case *6* **thus** *?case* **by** (*simp add*: *avl_insert*)
next
  case *7* **thus** *?case* **by** (*simp add*: *avl_delete*)
qed

end

19 AVL Tree with Balance Factors (2)

theory *AVL_Bal2_Set*
imports
  Cmp
  Isin2
begin

This version passes a flag (*Same*/*Diff*) back up to signal if the height changed.

datatype *bal* = *Lh* | *Bal* | *Rh*

type_synonym *'a tree_bal* = (*'a* ∗ *bal*) *tree*

Invariant:

fun *avl* :: *'a tree_bal* ⇒ *bool* **where**
avl Leaf = *True* |
avl (*Node l* (*a*,*b*) *r*) =
  ((*case b of*

```
    Bal ⇒ height r = height l |
    Lh ⇒ height l = height r + 1 |
    Rh ⇒ height r = height l + 1)
  ∧ avl l ∧ avl r)
```

19.1 Code

```
datatype 'a alt = Same 'a | Diff 'a

type_synonym 'a tree_bal2 = 'a tree_bal alt

fun tree :: 'a alt ⇒ 'a where
tree(Same t) = t |
tree(Diff t) = t

fun rot2 where
rot2 A a B c C = (case B of
  (Node B₁ (b, bb) B₂) ⇒
    let b₁ = if bb = Rh then Lh else Bal;
        b₂ = if bb = Lh then Rh else Bal
    in Node (Node A (a,b₁) B₁) (b,Bal) (Node B₂ (c,b₂) C))

fun balL :: 'a tree_bal2 ⇒ 'a ⇒ bal ⇒ 'a tree_bal ⇒ 'a tree_bal2 where
balL AB' c bc C = (case AB' of
   Same AB ⇒ Same (Node AB (c,bc) C) |
   Diff AB ⇒ (case bc of
     Bal ⇒ Diff (Node AB (c,Lh) C) |
     Rh ⇒ Same (Node AB (c,Bal) C) |
     Lh ⇒ (case AB of
       Node A (a,Lh) B ⇒ Same(Node A (a,Bal) (Node B (c,Bal) C)) |
       Node A (a,Rh) B ⇒ Same(rot2 A a B c C))))

fun balR :: 'a tree_bal ⇒ 'a ⇒ bal ⇒ 'a tree_bal2 ⇒ 'a tree_bal2 where
balR A a ba BC' = (case BC' of
   Same BC ⇒ Same (Node A (a,ba) BC) |
   Diff BC ⇒ (case ba of
     Bal ⇒ Diff (Node A (a,Rh) BC) |
     Lh ⇒ Same (Node A (a,Bal) BC) |
     Rh ⇒ (case BC of
       Node B (c,Rh) C ⇒ Same(Node (Node A (a,Bal) B) (c,Bal) C) |
       Node B (c,Lh) C ⇒ Same(rot2 A a B c C))))

fun ins :: 'a::linorder ⇒ 'a tree_bal ⇒ 'a tree_bal2 where
ins x Leaf = Diff(Node Leaf (x, Bal) Leaf) |
```

ins x (Node l (a, b) r) = (case cmp x a of
EQ ⇒ Same(Node l (a, b) r) |
LT ⇒ balL (ins x l) a b r |
GT ⇒ balR l a b (ins x r))

definition *insert* :: *′a::linorder ⇒ ′a tree_bal ⇒ ′a tree_bal* **where**
insert x t = tree(ins x t)

fun *baldR* :: *′a tree_bal ⇒ ′a ⇒ bal ⇒ ′a tree_bal2 ⇒ ′a tree_bal2* **where**
baldR AB c bc C′ = (case C′ of
Same C ⇒ Same (Node AB (c,bc) C) |
Diff C ⇒ (case bc of
Bal ⇒ Same (Node AB (c,Lh) C) |
Rh ⇒ Diff (Node AB (c,Bal) C) |
Lh ⇒ (case AB of
Node A (a,Lh) B ⇒ Diff(Node A (a,Bal) (Node B (c,Bal) C)) |
Node A (a,Bal) B ⇒ Same(Node A (a,Rh) (Node B (c,Lh) C)) |
Node A (a,Rh) B ⇒ Diff(rot2 A a B c C))))

fun *baldL* :: *′a tree_bal2 ⇒ ′a ⇒ bal ⇒ ′a tree_bal ⇒ ′a tree_bal2* **where**
baldL A′ a ba BC = (case A′ of
Same A ⇒ Same (Node A (a,ba) BC) |
Diff A ⇒ (case ba of
Bal ⇒ Same (Node A (a,Rh) BC) |
Lh ⇒ Diff (Node A (a,Bal) BC) |
Rh ⇒ (case BC of
Node B (c,Rh) C ⇒ Diff(Node (Node A (a,Bal) B) (c,Bal) C) |
Node B (c,Bal) C ⇒ Same(Node (Node A (a,Rh) B) (c,Lh) C) |
Node B (c,Lh) C ⇒ Diff(rot2 A a B c C))))

fun *split_max* :: *′a tree_bal ⇒ ′a tree_bal2 ∗ ′a* **where**
split_max (Node l (a, ba) r) =
(if r = Leaf then (Diff l,a) else let (r′,a′) = split_max r in (baldR l a ba r′, a′))

fun *del* :: *′a::linorder ⇒ ′a tree_bal ⇒ ′a tree_bal2* **where**
del _ Leaf = Same Leaf |
del x (Node l (a, ba) r) =
(case cmp x a of
EQ ⇒ if l = Leaf then Diff r
else let (l′, a′) = split_max l in baldL l′ a′ ba r |
LT ⇒ baldL (del x l) a ba r |
GT ⇒ baldR l a ba (del x r))

definition *delete* :: 'a::linorder ⇒ 'a tree_bal ⇒ 'a tree_bal **where**
delete x t = tree(del x t)

lemmas *split_max_induct = split_max.induct[case_names Node Leaf]*

lemmas *splits = if_splits tree.splits alt.splits bal.splits*

19.2 Proofs

19.2.1 Proofs about insertion

lemma *avl_ins_case*: *avl t ⟹ case ins x t of*
  Same t' ⇒ avl t' ∧ height t' = height t |
  Diff t' ⇒ avl t' ∧ height t' = height t + 1 ∧
    (∀ l a r. t' = Node l (a,Bal) r ⟶ a = x ∧ l = Leaf ∧ r = Leaf)
apply(*induction x t rule*: *ins.induct*)
apply(*auto simp*: *max_absorb1 split*!: *splits*)
done

corollary *avl_insert*: *avl t ⟹ avl(insert x t)*
using *avl_ins_case[of t x]* **by** (*simp add*: *insert_def split*: *splits*)

lemma *ins_Diff[simp]*: *avl t ⟹*
  ins x t ≠ Diff Leaf ∧
  (ins x t = Diff (Node l (a,Bal) r) ⟷ t = Leaf ∧ a = x ∧ l=Leaf ∧ r=Leaf) ∧
  ins x t ≠ Diff (Node l (a,Rh) Leaf) ∧
  ins x t ≠ Diff (Node Leaf (a,Lh) r)
by(*drule avl_ins_case[of _ x]*) (*auto split*: *splits*)

theorem *inorder_ins*:
  ⟦ *avl t*; *sorted(inorder t)* ⟧ ⟹ *inorder(tree(ins x t)) = ins_list x (inorder t)*
apply(*induction t*)
apply (*auto simp*: *ins_list_simps split*!: *splits*)
done

19.2.2 Proofs about deletion

lemma *inorder_baldL*:
  ⟦ *ba = Rh ⟶ r ≠ Leaf*; *avl r* ⟧
  ⟹ *inorder (tree(baldL l a ba r)) = inorder (tree l) @ a # inorder r*
by (*auto split*: *splits*)

lemma *inorder_baldR*:
⟦ *ba* = *Lh* ⟶ *l* ≠ *Leaf*; *avl l* ⟧
⟹ *inorder* (*tree*(*baldR l a ba r*)) = *inorder l* @ *a* # *inorder* (*tree r*)
by (*auto split*: *splits*)

lemma *avl_split_max*:
⟦ *split_max t* = (*t*′,*a*); *avl t*; *t* ≠ *Leaf* ⟧ ⟹ *case t*′ *of*
Same t′ ⇒ *avl t*′ ∧ *height t* = *height t*′ |
Diff t′ ⇒ *avl t*′ ∧ *height t* = *height t*′ + *1*
apply(*induction t arbitrary*: *t*′ *a rule*: *split_max_induct*)
apply(*fastforce simp*: *max_absorb1 max_absorb2 split*!: *splits prod.splits*)
apply *simp*
done

lemma *avl_del_case*: *avl t* ⟹ *case del x t of*
Same t′ ⇒ *avl t*′ ∧ *height t* = *height t*′ |
Diff t′ ⇒ *avl t*′ ∧ *height t* = *height t*′ + *1*
apply(*induction x t rule*: *del.induct*)
apply(*auto simp*: *max_absorb1 max_absorb2 dest*: *avl_split_max split*!: *splits prod.splits*)
done

corollary *avl_delete*: *avl t* ⟹ *avl*(*delete x t*)
using *avl_del_case*[*of t x*] **by**(*simp add*: *delete_def split*: *splits*)

lemma *inorder_split_maxD*:
⟦ *split_max t* = (*t*′,*a*); *t* ≠ *Leaf*; *avl t* ⟧ ⟹
inorder (*tree t*′) @ [*a*] = *inorder t*
apply(*induction t arbitrary*: *t*′ *rule*: *split_max.induct*)
apply(*fastforce split*!: *splits prod.splits*)
apply *simp*
done

lemma *neq_Leaf_if_height_neq_0*[*simp*]: *height t* ≠ *0* ⟹ *t* ≠ *Leaf*
by *auto*

theorem *inorder_del*:
⟦ *avl t*; *sorted*(*inorder t*) ⟧ ⟹ *inorder* (*tree*(*del x t*)) = *del_list x* (*inorder t*)
apply(*induction t rule*: *tree2_induct*)
apply(*auto simp*: *del_list_simps inorder_baldL inorder_baldR avl_delete inorder_split_maxD*
simp del: *baldR.simps baldL.simps split*!: *splits prod.splits*)

done

19.2.3 Set Implementation

```
interpretation S: Set_by_Ordered
where empty = Leaf and isin = isin
  and insert = insert
  and delete = delete
  and inorder = inorder and inv = avl
proof (standard, goal_cases)
  case 1 show ?case by (simp)
next
  case 2 thus ?case by(simp add: isin_set_inorder)
next
  case 3 thus ?case by(simp add: inorder_ins insert_def)
next
  case 4 thus ?case by(simp add: inorder_del delete_def)
next
  case 5 thus ?case by (simp)
next
  case 6 thus ?case by (simp add: avl_insert)
next
  case 7 thus ?case by (simp add: avl_delete)
qed

end
```

20 Height-Balanced Trees

```
theory Height_Balanced_Tree
imports
  Cmp
  Isin2
begin
```

Height-balanced trees (HBTs) can be seen as a generalization of AVL trees. The code and the proofs were obtained by small modifications of the AVL theories. This is an implementation of sets via HBTs.

```
type_synonym 'a tree_ht = ('a∗nat) tree

definition empty :: 'a tree_ht where
empty = Leaf
```

The maximal amount by which the height of two siblings may differ:

locale *HBT* =
fixes *m* :: *nat*
assumes [*arith*]: *m* > *0*
begin

Invariant:

fun *hbt* :: *'a tree_ht* ⇒ *bool* **where**
hbt Leaf = *True* |
hbt (*Node l* (*a*,*n*) *r*) =
 (*abs*(*int*(*height l*) − *int*(*height r*)) ≤ *int*(*m*) ∧
  n = *max* (*height l*) (*height r*) + *1* ∧ *hbt l* ∧ *hbt r*)

fun *ht* :: *'a tree_ht* ⇒ *nat* **where**
ht Leaf = *0* |
ht (*Node l* (*a*,*n*) *r*) = *n*

definition *node* :: *'a tree_ht* ⇒ *'a* ⇒ *'a tree_ht* ⇒ *'a tree_ht* **where**
node l a r = *Node l* (*a*, *max* (*ht l*) (*ht r*) + *1*) *r*

definition *balL* :: *'a tree_ht* ⇒ *'a* ⇒ *'a tree_ht* ⇒ *'a tree_ht* **where**
balL AB b C =
  (*if ht AB* = *ht C* + *m* + *1 then*
    case AB of
      Node A (*a*, _) *B* ⇒
        if ht A ≥ *ht B then node A a* (*node B b C*)
        else
          case B of
            Node B_1 (*ab*, _) B_2 ⇒ *node* (*node A a* B_1) *ab* (*node* B_2 *b C*)
  else node AB b C)

definition *balR* :: *'a tree_ht* ⇒ *'a* ⇒ *'a tree_ht* ⇒ *'a tree_ht* **where**
balR A a BC =
  (*if ht BC* = *ht A* + *m* + *1 then*
    case BC of
      Node B (*b*, _) *C* ⇒
        if ht B ≤ *ht C then node* (*node A a B*) *b C*
        else
          case B of
            Node B_1 (*ab*, _) B_2 ⇒ *node* (*node A a* B_1) *ab* (*node* B_2 *b C*)
  else node A a BC)

fun *insert* :: *'a::linorder* ⇒ *'a tree_ht* ⇒ *'a tree_ht* **where**
insert x Leaf = *Node Leaf* (*x*, *1*) *Leaf* |
insert x (*Node l* (*a*, *n*) *r*) = (*case cmp x a of*

```
  EQ ⇒ Node l (a, n) r |
  LT ⇒ balL (insert x l) a r |
  GT ⇒ balR l a (insert x r))
```

```
fun split_max :: 'a tree_ht ⇒ 'a tree_ht * 'a where
split_max (Node l (a, _) r) =
  (if r = Leaf then (l,a) else let (r',a') = split_max r in (balL l a r', a'))
```

```
lemmas split_max_induct = split_max.induct[case_names Node Leaf]
```

```
fun delete :: 'a::linorder ⇒ 'a tree_ht ⇒ 'a tree_ht where
delete _ Leaf = Leaf |
delete x (Node l (a, n) r) =
  (case cmp x a of
     EQ ⇒ if l = Leaf then r
          else let (l', a') = split_max l in balR l' a' r |
     LT ⇒ balR (delete x l) a r |
     GT ⇒ balL l a (delete x r))
```

20.1 Functional Correctness Proofs

20.1.1 Proofs for insert

```
lemma inorder_balL:
  inorder (balL l a r) = inorder l @ a # inorder r
by (auto simp: node_def balL_def split:tree.splits)
```

```
lemma inorder_balR:
  inorder (balR l a r) = inorder l @ a # inorder r
by (auto simp: node_def balR_def split:tree.splits)
```

```
theorem inorder_insert:
  sorted(inorder t) ⟹ inorder(insert x t) = ins_list x (inorder t)
by (induct t)
   (auto simp: ins_list_simps inorder_balL inorder_balR)
```

20.1.2 Proofs for delete

```
lemma inorder_split_maxD:
  ⟦ split_max t = (t',a); t ≠ Leaf ⟧ ⟹
   inorder t' @ [a] = inorder t
by(induction t arbitrary: t' rule: split_max.induct)
  (auto simp: inorder_balL split: if_splits prod.splits tree.split)
```

```
theorem inorder_delete:
```

sorted(*inorder t*) ⟹ *inorder* (*delete x t*) = *del_list x* (*inorder t*)
by(*induction t*)
(*auto simp*: *del_list_simps inorder_balL inorder_balR inorder_split_maxD split*: *prod.splits*)

20.2 Invariant preservation

20.2.1 Insertion maintains balance

declare *Let_def* [*simp*]

lemma *ht_height*[*simp*]: *hbt t* ⟹ *ht t* = *height t*
by (*cases t rule*: *tree2_cases*) *simp_all*

First, a fast but relatively manual proof with many lemmas:

lemma *height_balL*:
⟦ *hbt l*; *hbt r*; *height l* = *height r* + *m* + *1* ⟧ ⟹
height (*balL l a r*) ∈ {*height r* + *m* + *1*, *height r* + *m* + *2*}
by (*auto simp*:*node_def balL_def split*:*tree.split*)

lemma *height_balR*:
⟦ *hbt l*; *hbt r*; *height r* = *height l* + *m* + *1* ⟧ ⟹
height (*balR l a r*) ∈ {*height l* + *m* + *1*, *height l* + *m* + *2*}
by(*auto simp add*:*node_def balR_def split*:*tree.split*)

lemma *height_node*[*simp*]: *height*(*node l a r*) = *max* (*height l*) (*height r*) + *1*
by (*simp add*: *node_def*)

lemma *height_balL2*:
⟦ *hbt l*; *hbt r*; *height l* ≠ *height r* + *m* + *1* ⟧ ⟹
height (*balL l a r*) = *1* + *max* (*height l*) (*height r*)
by (*simp_all add*: *balL_def*)

lemma *height_balR2*:
⟦ *hbt l*; *hbt r*; *height r* ≠ *height l* + *m* + *1* ⟧ ⟹
height (*balR l a r*) = *1* + *max* (*height l*) (*height r*)
by (*simp_all add*: *balR_def*)

lemma *hbt_balL*:
⟦ *hbt l*; *hbt r*; *height r* − *m* ≤ *height l* ∧ *height l* ≤ *height r* + *m* + *1* ⟧ ⟹ *hbt*(*balL l a r*)
by(*auto simp*: *balL_def node_def max_def split*!: *if_splits tree.split*)

lemma *hbt_balR*:

⟦ *hbt l*; *hbt r*; *height l* − *m* ≤ *height r* ∧ *height r* ≤ *height l* + *m* + *1* ⟧
⟹ *hbt*(*balR l a r*)
by(*auto simp*: *balR_def node_def max_def split*!: *if_splits tree.split*)

Insertion maintains *hbt*. Requires simultaneous proof.

theorem *hbt_insert*:
  hbt t ⟹ *hbt*(*insert x t*)
  hbt t ⟹ *height* (*insert x t*) ∈ {*height t*, *height t* + *1*}
proof (*induction t rule*: *tree2_induct*)
  case (*Node l a _ r*)
  case *1*
  show *?case*
  proof(*cases x* = *a*)
    case *True* **with** *Node 1* **show** *?thesis* **by** *simp*
  next
    case *False*
    show *?thesis*
    proof(*cases x*<*a*)
      case *True* **with** *1 Node*(*1*,*2*) **show** *?thesis* **by** (*auto intro*!: *hbt_balL*)
    next
      case *False* **with** *1 Node*(*3*,*4*) ‹*x*≠*a*› **show** *?thesis* **by** (*auto intro*!: *hbt_balR*)
    qed
  qed
  case *2*
  show *?case*
  proof(*cases x* = *a*)
    case *True* **with** *2* **show** *?thesis* **by** *simp*
  next
    case *False*
    show *?thesis*
    proof(*cases x*<*a*)
      case *True*
      show *?thesis*
      proof(*cases height* (*insert x l*) = *height r* + *m* + *1*)
        case *False* **with** *2 Node*(*1*,*2*) ‹*x* < *a*› **show** *?thesis* **by** (*auto simp*: *height_balL2*)
      next
        case *True*
        hence (*height* (*balL* (*insert x l*) *a r*) = *height r* + *m* + *1*) ∨
          (*height* (*balL* (*insert x l*) *a r*) = *height r* + *m* + *2*) (**is** *?A* ∨ *?B*)
          using *2 Node*(*1*,*2*) *height_balL*[*OF _ _ True*] **by** *simp*
        thus *?thesis*
        proof

assume *?A* **with** *2 Node(2) True* ‹*x* < *a*› **show** *?thesis* **by** (*auto*)
next
assume *?B* **with** *2 Node(2) True* ‹*x* < *a*› **show** *?thesis* **by** (*simp*) *arith*
qed
qed
next
case *False*
show *?thesis*
proof(*cases height* (*insert x r*) = *height l* + *m* + *1*)
case *False* **with** *2 Node(3,4)* ‹¬*x* < *a*› **show** *?thesis* **by** (*auto simp*: *height_balR2*)
next
case *True*
hence (*height* (*balR l a* (*insert x r*)) = *height l* + *m* + *1*) ∨
(*height* (*balR l a* (*insert x r*)) = *height l* + *m* + *2*) (**is** *?A* ∨ *?B*)
using *Node 2 height_balR*[*OF _ _ True*] **by** *simp*
thus *?thesis*
proof
assume *?A* **with** *2 Node(4) True* ‹¬*x* < *a*› **show** *?thesis* **by** (*auto*)
next
assume *?B* **with** *2 Node(4) True* ‹¬*x* < *a*› **show** *?thesis* **by** (*simp*) *arith*
qed
qed
qed
qed
qed *simp_all*

Now an automatic proof without lemmas:

theorem *hbt_insert_auto*: *hbt t* ⟹
hbt(*insert x t*) ∧ *height* (*insert x t*) ∈ {*height t*, *height t* + *1*}
apply (*induction t rule*: *tree2_induct*)

apply (*auto simp*: *balL_def balR_def node_def max_absorb1 max_absorb2 split*!: *if_split tree.split*)
done

20.2.2 Deletion maintains balance

lemma *hbt_split_max*:
⟦ *hbt t*; *t* ≠ *Leaf* ⟧ ⟹
hbt (*fst* (*split_max t*)) ∧
height t ∈ {*height*(*fst* (*split_max t*)), *height*(*fst* (*split_max t*)) + *1*}

by(*induct t rule*: *split_max_induct*)
(*auto simp*: *balL_def node_def max_absorb2 split*!: *prod.split if_split tree.split*)

Deletion maintains *hbt*:

theorem *hbt_delete*:
hbt t ⟹ *hbt*(*delete x t*)
hbt t ⟹ *height t* ∈ {*height* (*delete x t*), *height* (*delete x t*) + *1*}
proof (*induct t rule*: *tree2_induct*)
case (*Node l a n r*)
case *1*
thus *?case*
using *Node hbt_split_max*[*of l*] **by** (*auto intro*!: *hbt_balL hbt_balR split*: *prod.split*)
case *2*
show *?case*
proof(*cases x* = *a*)
case *True* **then show** *?thesis* **using** *1 hbt_split_max*[*of l*]
by(*auto simp*: *balR_def max_absorb2 split*!: *if_splits prod.split tree.split*)
next
case *False*
show *?thesis*
proof(*cases x*<*a*)
case *True*
show *?thesis*
proof(*cases height r* = *height* (*delete x l*) + *m* + *1*)
case *False* **with** *Node 1* ‹*x* < *a*› **show** *?thesis* **by**(*auto simp*: *balR_def*)
next
case *True*
hence (*height* (*balR* (*delete x l*) *a r*) = *height* (*delete x l*) + *m* + *1*) ∨
height (*balR* (*delete x l*) *a r*) = *height* (*delete x l*) + *m* + *2* (**is** *?A* ∨ *?B*)
using *Node 2height_balR*[*OF* _ _ *True*] **by** *simp*
thus *?thesis*
proof
assume *?A* **with** ‹*x* < *a*› *Node 2* **show** *?thesis* **by**(*auto simp*: *balR_def split*!: *if_splits*)
next
assume *?B* **with** ‹*x* < *a*› *Node 2* **show** *?thesis* **by**(*auto simp*: *balR_def split*!: *if_splits*)
qed
qed

next
case *False*
show *?thesis*
proof(*cases height l = height (delete x r) + m + 1*)
case *False* **with** *Node 1* ‹¬*x < a*› ‹*x* ≠ *a*› **show** *?thesis* **by**(*auto simp*: *balL_def*)
next
case *True*
hence (*height* (*balL l a* (*delete x r*)) = *height* (*delete x r*) + *m* + *1*) ∨
height (*balL l a* (*delete x r*)) = *height* (*delete x r*) + *m* + *2* (**is** *?A* ∨ *?B*)
using *Node 2 height_balL*[*OF _ _ True*] **by** *simp*
thus *?thesis*
proof
assume *?A* **with** ‹¬*x < a*› ‹*x* ≠ *a*› *Node 2* **show** *?thesis* **by**(*auto simp*: *balL_def split*: *if_splits*)
next
assume *?B* **with** ‹¬*x < a*› ‹*x* ≠ *a*› *Node 2* **show** *?thesis* **by**(*auto simp*: *balL_def split*: *if_splits*)
qed
qed
qed
qed
qed *simp_all*

A more automatic proof. Complete automation as for insertion seems hard due to resource requirements.

theorem *hbt_delete_auto*:
hbt t ⟹ *hbt*(*delete x t*)
hbt t ⟹ *height t* ∈ {*height* (*delete x t*), *height* (*delete x t*) + *1*}
proof (*induct t rule*: *tree2_induct*)
case (*Node l a n r*)
case *1*
thus *?case*
using *Node hbt_split_max*[*of l*] **by** (*auto intro*!: *hbt_balL hbt_balR split*: *prod.split*)
case *2*
show *?case*
proof(*cases x = a*)
case *True* **thus** *?thesis*
using *2 hbt_split_max*[*of l*]
by(*auto simp*: *balR_def max_absorb2 split*!: *if_splits prod.split tree.split*)
next

case *False* **thus** *?thesis*
using *height_balL*[*of l delete x r a*] *height_balR*[*of delete x l r a*] *2 Node*
by(*auto simp*: *balL_def balR_def split*!: *if_split*)
qed
qed *simp_all*

20.3 Overall correctness

interpretation *S*: *Set_by_Ordered*
where *empty* = *empty* **and** *isin* = *isin* **and** *insert* = *insert* **and** *delete* = *delete*
and *inorder* = *inorder* **and** *inv* = *hbt*
proof (*standard*, *goal_cases*)
case *1* **show** *?case* **by** (*simp add*: *empty_def*)
next
case *2* **thus** *?case* **by**(*simp add*: *isin_set_inorder*)
next
case *3* **thus** *?case* **by**(*simp add*: *inorder_insert*)
next
case *4* **thus** *?case* **by**(*simp add*: *inorder_delete*)
next
case *5* **thus** *?case* **by** (*simp add*: *empty_def*)
next
case *6* **thus** *?case* **by** (*simp add*: *hbt_insert*(*1*))
next
case *7* **thus** *?case* **by** (*simp add*: *hbt_delete*(*1*))
qed

end

end

21 Red-Black Trees

theory *RBT*
imports *Tree2*
begin

datatype *color* = *Red* | *Black*

type_synonym *'a rbt* = (*'a*∗*color*)*tree*

abbreviation *R* **where** *R l a r* ≡ *Node l* (*a*, *Red*) *r*

```
abbreviation B where B l a r ≡ Node l (a, Black) r

fun baliL :: 'a rbt ⇒ 'a ⇒ 'a rbt ⇒ 'a rbt where
baliL (R (R t1 a t2) b t3) c t4 = R (B t1 a t2) b (B t3 c t4) |
baliL (R t1 a (R t2 b t3)) c t4 = R (B t1 a t2) b (B t3 c t4) |
baliL t1 a t2 = B t1 a t2

fun baliR :: 'a rbt ⇒ 'a ⇒ 'a rbt ⇒ 'a rbt where
baliR t1 a (R t2 b (R t3 c t4)) = R (B t1 a t2) b (B t3 c t4) |
baliR t1 a (R (R t2 b t3) c t4) = R (B t1 a t2) b (B t3 c t4) |
baliR t1 a t2 = B t1 a t2

fun paint :: color ⇒ 'a rbt ⇒ 'a rbt where
paint c Leaf = Leaf |
paint c (Node l (a,_) r) = Node l (a,c) r

fun baldL :: 'a rbt ⇒ 'a ⇒ 'a rbt ⇒ 'a rbt where
baldL (R t1 a t2) b t3 = R (B t1 a t2) b t3 |
baldL t1 a (B t2 b t3) = baliR t1 a (R t2 b t3) |
baldL t1 a (R (B t2 b t3) c t4) = R (B t1 a t2) b (baliR t3 c (paint Red
t4)) |
baldL t1 a t2 = R t1 a t2

fun baldR :: 'a rbt ⇒ 'a ⇒ 'a rbt ⇒ 'a rbt where
baldR t1 a (R t2 b t3) = R t1 a (B t2 b t3) |
baldR (B t1 a t2) b t3 = baliL (R t1 a t2) b t3 |
baldR (R t1 a (B t2 b t3)) c t4 = R (baliL (paint Red t1) a t2) b (B t3 c
t4) |
baldR t1 a t2 = R t1 a t2

fun join :: 'a rbt ⇒ 'a rbt ⇒ 'a rbt where
join Leaf t = t |
join t Leaf = t |
join (R t1 a t2) (R t3 c t4) =
  (case join t2 t3 of
     R u2 b u3 ⇒ (R (R t1 a u2) b (R u3 c t4)) |
     t23 ⇒ R t1 a (R t23 c t4)) |
join (B t1 a t2) (B t3 c t4) =
  (case join t2 t3 of
     R u2 b u3 ⇒ R (B t1 a u2) b (B u3 c t4) |
     t23 ⇒ baldL t1 a (B t23 c t4)) |
join t1 (R t2 a t3) = R (join t1 t2) a t3 |
join (R t1 a t2) t3 = R t1 a (join t2 t3)
```

end

22 Red-Black Tree Implementation of Sets

```
theory RBT_Set
imports
  Complex_Main
  RBT
  Cmp
  Isin2
begin

definition empty :: 'a rbt where
empty = Leaf

fun ins :: 'a::linorder ⇒ 'a rbt ⇒ 'a rbt where
ins x Leaf = R Leaf x Leaf |
ins x (B l a r) =
  (case cmp x a of
     LT ⇒ baliL (ins x l) a r |
     GT ⇒ baliR l a (ins x r) |
     EQ ⇒ B l a r) |
ins x (R l a r) =
  (case cmp x a of
    LT ⇒ R (ins x l) a r |
    GT ⇒ R l a (ins x r) |
    EQ ⇒ R l a r)

definition insert :: 'a::linorder ⇒ 'a rbt ⇒ 'a rbt where
insert x t = paint Black (ins x t)

fun color :: 'a rbt ⇒ color where
color Leaf = Black |
color (Node _ (_, c) _) = c

fun del :: 'a::linorder ⇒ 'a rbt ⇒ 'a rbt where
del x Leaf = Leaf |
del x (Node l (a, _) r) =
  (case cmp x a of
     LT ⇒ if l ≠ Leaf ∧ color l = Black
          then baldL (del x l) a r else R (del x l) a r |
     GT ⇒ if r ≠ Leaf∧ color r = Black
          then baldR l a (del x r) else R l a (del x r) |
```

EQ ⇒ *join l r*)

definition *delete* :: *'a*::*linorder* ⇒ *'a rbt* ⇒ *'a rbt* **where**
delete x t = *paint Black* (*del x t*)

22.1 Functional Correctness Proofs

lemma *inorder_paint*: *inorder*(*paint c t*) = *inorder t*
by(*cases t*) (*auto*)

lemma *inorder_baliL*:
inorder(*baliL l a r*) = *inorder l* @ *a* # *inorder r*
by(*cases* (*l*,*a*,*r*) *rule*: *baliL.cases*) (*auto*)

lemma *inorder_baliR*:
inorder(*baliR l a r*) = *inorder l* @ *a* # *inorder r*
by(*cases* (*l*,*a*,*r*) *rule*: *baliR.cases*) (*auto*)

lemma *inorder_ins*:
sorted(*inorder t*) ⟹ *inorder*(*ins x t*) = *ins_list x* (*inorder t*)
by(*induction x t rule*: *ins.induct*)
(*auto simp*: *ins_list_simps inorder_baliL inorder_baliR*)

lemma *inorder_insert*:
sorted(*inorder t*) ⟹ *inorder*(*insert x t*) = *ins_list x* (*inorder t*)
by (*simp add*: *insert_def inorder_ins inorder_paint*)

lemma *inorder_baldL*:
inorder(*baldL l a r*) = *inorder l* @ *a* # *inorder r*
by(*cases* (*l*,*a*,*r*) *rule*: *baldL.cases*)
(*auto simp*: *inorder_baliL inorder_baliR inorder_paint*)

lemma *inorder_baldR*:
inorder(*baldR l a r*) = *inorder l* @ *a* # *inorder r*
by(*cases* (*l*,*a*,*r*) *rule*: *baldR.cases*)
(*auto simp*: *inorder_baliL inorder_baliR inorder_paint*)

lemma *inorder_join*:
inorder(*join l r*) = *inorder l* @ *inorder r*
by(*induction l r rule*: *join.induct*)
(*auto simp*: *inorder_baldL inorder_baldR split*: *tree.split color.split*)

lemma *inorder_del*:
sorted(*inorder t*) ⟹ *inorder*(*del x t*) = *del_list x* (*inorder t*)

by(*induction x t rule*: *del.induct*)
(*auto simp*: *del_list_simps inorder_join inorder_baldL inorder_baldR*)

lemma *inorder_delete*:
sorted(*inorder t*) $\Longrightarrow$ *inorder*(*delete x t*) = *del_list x* (*inorder t*)
by (*auto simp*: *delete_def inorder_del inorder_paint*)

22.2 Structural invariants

lemma *neq_Black*[*simp*]: (*c* $\neq$ *Black*) = (*c* = *Red*)
by (*cases c*) *auto*

The proofs are due to Markus Reiter and Alexander Krauss.

fun *bheight* :: *'a rbt* $\Rightarrow$ *nat* **where**
bheight Leaf = *0* |
bheight (*Node l* (*x*, *c*) *r*) = (*if c* = *Black then bheight l* + *1 else bheight l*)

fun *invc* :: *'a rbt* $\Rightarrow$ *bool* **where**
invc Leaf = *True* |
invc (*Node l* (*a*,*c*) *r*) =
((*c* = *Red* $\longrightarrow$ *color l* = *Black* $\wedge$ *color r* = *Black*) $\wedge$ *invc l* $\wedge$ *invc r*)

Weaker version:

abbreviation *invc2* :: *'a rbt* $\Rightarrow$ *bool* **where**
invc2 t $\equiv$ *invc*(*paint Black t*)

fun *invh* :: *'a rbt* $\Rightarrow$ *bool* **where**
invh Leaf = *True* |
invh (*Node l* (*x*, *c*) *r*) = (*bheight l* = *bheight r* $\wedge$ *invh l* $\wedge$ *invh r*)

lemma *invc2I*: *invc t* $\Longrightarrow$ *invc2 t*
by (*cases t rule*: *tree2_cases*) *simp+*

definition *rbt* :: *'a rbt* $\Rightarrow$ *bool* **where**
rbt t = (*invc t* $\wedge$ *invh t* $\wedge$ *color t* = *Black*)

lemma *color_paint_Black*: *color* (*paint Black t*) = *Black*
by (*cases t*) *auto*

lemma *paint2*: *paint c2* (*paint c1 t*) = *paint c2 t*
by (*cases t*) *auto*

lemma *invh_paint*: *invh t* $\Longrightarrow$ *invh* (*paint c t*)
by (*cases t*) *auto*

lemma *invc_baliL*:
  $[\![invc2\ l;\ invc\ r]\!] \Longrightarrow invc\ (baliL\ l\ a\ r)$
by (*induct l a r rule*: *baliL.induct*) *auto*

lemma *invc_baliR*:
  $[\![invc\ l;\ invc2\ r]\!] \Longrightarrow invc\ (baliR\ l\ a\ r)$
by (*induct l a r rule*: *baliR.induct*) *auto*

lemma *bheight_baliL*:
  $bheight\ l = bheight\ r \Longrightarrow bheight\ (baliL\ l\ a\ r) = Suc\ (bheight\ l)$
by (*induct l a r rule*: *baliL.induct*) *auto*

lemma *bheight_baliR*:
  $bheight\ l = bheight\ r \Longrightarrow bheight\ (baliR\ l\ a\ r) = Suc\ (bheight\ l)$
by (*induct l a r rule*: *baliR.induct*) *auto*

lemma *invh_baliL*:
  $[\![\ invh\ l;\ invh\ r;\ bheight\ l = bheight\ r\ ]\!] \Longrightarrow invh\ (baliL\ l\ a\ r)$
by (*induct l a r rule*: *baliL.induct*) *auto*

lemma *invh_baliR*:
  $[\![\ invh\ l;\ invh\ r;\ bheight\ l = bheight\ r\ ]\!] \Longrightarrow invh\ (baliR\ l\ a\ r)$
by (*induct l a r rule*: *baliR.induct*) *auto*

All in one:

lemma *inv_baliR*: $[\![\ invh\ l;\ invh\ r;\ invc\ l;\ invc2\ r;\ bheight\ l = bheight\ r\ ]\!] \Longrightarrow invc\ (baliR\ l\ a\ r) \wedge invh\ (baliR\ l\ a\ r) \wedge bheight\ (baliR\ l\ a\ r) = Suc\ (bheight\ l)$
by (*induct l a r rule*: *baliR.induct*) *auto*

lemma *inv_baliL*: $[\![\ invh\ l;\ invh\ r;\ invc2\ l;\ invc\ r;\ bheight\ l = bheight\ r\ ]\!] \Longrightarrow invc\ (baliL\ l\ a\ r) \wedge invh\ (baliL\ l\ a\ r) \wedge bheight\ (baliL\ l\ a\ r) = Suc\ (bheight\ l)$
by (*induct l a r rule*: *baliL.induct*) *auto*

22.2.1 Insertion

lemma *invc_ins*: $invc\ t \longrightarrow invc2\ (ins\ x\ t) \wedge (color\ t = Black \longrightarrow invc\ (ins\ x\ t))$
by (*induct x t rule*: *ins.induct*) (*auto simp*: *invc_baliL invc_baliR invc2I*)

lemma *invh_ins*: $invh\ t \Longrightarrow invh\ (ins\ x\ t) \wedge bheight\ (ins\ x\ t) = bheight\ t$
by(*induct x t rule*: *ins.induct*)

(*auto simp*: *invh_baliL invh_baliR bheight_baliL bheight_baliR*)

theorem *rbt_insert*: *rbt t* $\Longrightarrow$ *rbt* (*insert x t*)
by (*simp add*: *invc_ins invh_ins color_paint_Black invh_paint rbt_def insert_def*)

All in one:

lemma *inv_ins*: $[\![$ *invc t*; *invh t* $]\!] \Longrightarrow$
invc2 (*ins x t*) $\wedge$ (*color t* = *Black* $\longrightarrow$ *invc* (*ins x t*)) $\wedge$
invh(*ins x t*) $\wedge$ *bheight* (*ins x t*) = *bheight t*
by (*induct x t rule*: *ins.induct*) (*auto simp*: *inv_baliL inv_baliR invc2I*)

theorem *rbt_insert2*: *rbt t* $\Longrightarrow$ *rbt* (*insert x t*)
by (*simp add*: *inv_ins color_paint_Black invh_paint rbt_def insert_def*)

22.2.2 Deletion

lemma *bheight_paint_Red*:
color t = *Black* $\Longrightarrow$ *bheight* (*paint Red t*) = *bheight t* − *1*
by (*cases t*) *auto*

lemma *invh_baldL_invc*:
$[\![$ *invh l*; *invh r*; *bheight l* + *1* = *bheight r*; *invc r* $]\!]$
$\Longrightarrow$ *invh* (*baldL l a r*) $\wedge$ *bheight* (*baldL l a r*) = *bheight r*
by (*induct l a r rule*: *baldL.induct*)
(*auto simp*: *invh_baliR invh_paint bheight_baliR bheight_paint_Red*)

lemma *invh_baldL_Black*:
$[\![$ *invh l*; *invh r*; *bheight l* + *1* = *bheight r*; *color r* = *Black* $]\!]$
$\Longrightarrow$ *invh* (*baldL l a r*) $\wedge$ *bheight* (*baldL l a r*) = *bheight r*
by (*induct l a r rule*: *baldL.induct*) (*auto simp add*: *invh_baliR bheight_baliR*)

lemma *invc_baldL*: $[\![$*invc2 l*; *invc r*; *color r* = *Black*$]\!] \Longrightarrow$ *invc* (*baldL l a r*)
by (*induct l a r rule*: *baldL.induct*) (*simp_all add*: *invc_baliR*)

lemma *invc2_baldL*: $[\![$ *invc2 l*; *invc r* $]\!] \Longrightarrow$ *invc2* (*baldL l a r*)
by (*induct l a r rule*: *baldL.induct*) (*auto simp*: *invc_baliR paint2 invc2I*)

lemma *invh_baldR_invc*:
$[\![$ *invh l*; *invh r*; *bheight l* = *bheight r* + *1*; *invc l* $]\!]$
$\Longrightarrow$ *invh* (*baldR l a r*) $\wedge$ *bheight* (*baldR l a r*) = *bheight l*
by(*induct l a r rule*: *baldR.induct*)

(*auto simp*: *invh_baliL bheight_baliL invh_paint bheight_paint_Red*)

lemma *invc_baldR*: $[\![$*invc l*; *invc2 r*; *color l* = *Black*$]\!] \Longrightarrow$ *invc* (*baldR l a r*)
by (*induct l a r rule*: *baldR.induct*) (*simp_all add*: *invc_baliL*)

lemma *invc2_baldR*: $[\![$ *invc l*; *invc2 r* $]\!] \Longrightarrow$*invc2* (*baldR l a r*)
by (*induct l a r rule*: *baldR.induct*) (*auto simp*: *invc_baliL paint2 invc2I*)

lemma *invh_join*:
$[\![$ *invh l*; *invh r*; *bheight l* = *bheight r* $]\!]$
$\Longrightarrow$ *invh* (*join l r*) ∧ *bheight* (*join l r*) = *bheight l*
by (*induct l r rule*: *join.induct*)
(*auto simp*: *invh_baldL_Black split*: *tree.splits color.splits*)

lemma *invc_join*:
$[\![$ *invc l*; *invc r* $]\!] \Longrightarrow$
(*color l* = *Black* ∧ *color r* = *Black* ⟶ *invc* (*join l r*)) ∧ *invc2* (*join l r*)
by (*induct l r rule*: *join.induct*)
(*auto simp*: *invc_baldL invc2I split*: *tree.splits color.splits*)

All in one:

lemma *inv_baldL*:
$[\![$ *invh l*; *invh r*; *bheight l* + *1* = *bheight r*; *invc2 l*; *invc r* $]\!]$
$\Longrightarrow$ *invh* (*baldL l a r*) ∧ *bheight* (*baldL l a r*) = *bheight r*
∧ *invc2* (*baldL l a r*) ∧ (*color r* = *Black* ⟶ *invc* (*baldL l a r*))
by (*induct l a r rule*: *baldL.induct*)
(*auto simp*: *inv_baliR invh_paint bheight_baliR bheight_paint_Red paint2 invc2I*)

lemma *inv_baldR*:
$[\![$ *invh l*; *invh r*; *bheight l* = *bheight r* + *1*; *invc l*; *invc2 r* $]\!]$
$\Longrightarrow$ *invh* (*baldR l a r*) ∧ *bheight* (*baldR l a r*) = *bheight l*
∧ *invc2* (*baldR l a r*) ∧ (*color l* = *Black* ⟶ *invc* (*baldR l a r*))
by (*induct l a r rule*: *baldR.induct*)
(*auto simp*: *inv_baliL invh_paint bheight_baliL bheight_paint_Red paint2 invc2I*)

lemma *inv_join*:
$[\![$ *invh l*; *invh r*; *bheight l* = *bheight r*; *invc l*; *invc r* $]\!]$
$\Longrightarrow$ *invh* (*join l r*) ∧ *bheight* (*join l r*) = *bheight l*
∧ *invc2* (*join l r*) ∧ (*color l* = *Black* ∧ *color r* = *Black* ⟶ *invc* (*join l r*))
by (*induct l r rule*: *join.induct*)

(*auto simp*: *invh_baldL_Black inv_baldL invc2I split*: *tree.splits color.splits*)

lemma *neq_LeafD*: $t \neq Leaf \Longrightarrow \exists\, l\ x\ c\ r.\ t = Node\ l\ (x,c)\ r$
by(*cases t rule*: *tree2_cases*) *auto*

lemma *inv_del*: $[\![\ invh\ t;\ invc\ t\ ]\!] \Longrightarrow$
  $invh\ (del\ x\ t) \wedge$
  $(color\ t = Red \longrightarrow bheight\ (del\ x\ t) = bheight\ t \wedge invc\ (del\ x\ t)) \wedge$
  $(color\ t = Black \longrightarrow bheight\ (del\ x\ t) = bheight\ t - 1 \wedge invc2\ (del\ x\ t))$
by(*induct x t rule*: *del.induct*)
 (*auto simp*: *inv_baldL inv_baldR inv_join dest*!: *neq_LeafD*)

theorem *rbt_delete*: $rbt\ t \Longrightarrow rbt\ (delete\ x\ t)$
by (*metis delete_def rbt_def color_paint_Black inv_del invh_paint*)

Overall correctness:

interpretation *S*: *Set_by_Ordered*
where *empty* = *empty* **and** *isin* = *isin* **and** *insert* = *insert* **and** *delete* = *delete*
and *inorder* = *inorder* **and** *inv* = *rbt*
proof (*standard*, *goal_cases*)
 case *1* **show** *?case* **by** (*simp add*: *empty_def*)
next
 case *2* **thus** *?case* **by**(*simp add*: *isin_set_inorder*)
next
 case *3* **thus** *?case* **by**(*simp add*: *inorder_insert*)
next
 case *4* **thus** *?case* **by**(*simp add*: *inorder_delete*)
next
 case *5* **thus** *?case* **by** (*simp add*: *rbt_def empty_def*)
next
 case *6* **thus** *?case* **by** (*simp add*: *rbt_insert*)
next
 case *7* **thus** *?case* **by** (*simp add*: *rbt_delete*)
qed

22.3 Height-Size Relation

lemma *rbt_height_bheight_if*: $invc\ t \Longrightarrow invh\ t \Longrightarrow$
 $height\ t \leq 2 * bheight\ t + (if\ color\ t = Black\ then\ 0\ else\ 1)$
by(*induction t*) (*auto split*: *if_split_asm*)

lemma *rbt_height_bheight*: $rbt\ t \Longrightarrow height\ t\ /\ 2 \leq bheight\ t$
by(*auto simp*: *rbt_def dest*: *rbt_height_bheight_if*)

lemma *bheight_size_bound*: *invc t* ⟹ *invh t* ⟹ *2* ^ (*bheight t*) ≤ *size1 t*
by (*induction t*) *auto*

lemma *rbt_height_le*: **assumes** *rbt t* **shows** *height t* ≤ *2* ∗ *log 2* (*size1 t*)
proof −
  have *2 powr* (*height t* / *2*) ≤ *2 powr bheight t*
    using *rbt_height_bheight*[*OF assms*] **by** (*simp*)
  also have . . . ≤ *size1 t* **using** *assms*
    by (*simp add*: *powr_realpow bheight_size_bound rbt_def*)
  finally have *2 powr* (*height t* / *2*) ≤ *size1 t* **.**
  hence *height t* / *2* ≤ *log 2* (*size1 t*)
    by (*simp add*: *le_log_iff size1_size del*: *divide_le_eq_numeral1*(*1*))
  thus *?thesis* **by** *simp*
qed

end

23 Alternative Deletion in Red-Black Trees

theory *RBT_Set2*
imports *RBT_Set*
begin

This is a conceptually simpler version of deletion. Instead of the tricky *join* function this version follows the standard approach of replacing the deleted element (in function *del*) by the minimal element in its right subtree.

fun *split_min* :: *'a rbt* ⇒ *'a* × *'a rbt* **where**
split_min (*Node l* (*a*, *_*) *r*) =
  (*if l* = *Leaf then* (*a*,*r*)
   else let (*x*,*l'*) = *split_min l*
       in (*x*, *if color l* = *Black then baldL l' a r else R l' a r*))

fun *del* :: *'a*::*linorder* ⇒ *'a rbt* ⇒ *'a rbt* **where**
del x Leaf = *Leaf* |
del x (*Node l* (*a*, *_*) *r*) =
  (*case cmp x a of*
     LT ⇒ *let l'* = *del x l in if l* ≠ *Leaf* ∧ *color l* = *Black*
          then baldL l' a r else R l' a r |
     GT ⇒ *let r'* = *del x r in if r* ≠ *Leaf* ∧ *color r* = *Black*
          then baldR l a r' else R l a r' |
     EQ ⇒ *if r* = *Leaf then l else let* (*a'*,*r'*) = *split_min r in*
          if color r = *Black then baldR l a' r' else R l a' r'*)

The first two *let*s speed up the automatic proof of *inv_del* below.

definition *delete* :: *'a::linorder* ⇒ *'a rbt* ⇒ *'a rbt* **where**
delete x t = *paint Black* (*del x t*)

23.1 Functional Correctness Proofs

declare *Let_def*[*simp*]

lemma *split_minD*:
split_min t = (*x*,*t'*) ⟹ *t* ≠ *Leaf* ⟹ *x* # *inorder t'* = *inorder t*
by(*induction t arbitrary*: *t' rule*: *split_min.induct*)
(*auto simp*: *inorder_baldL sorted_lems split*: *prod.splits if_splits*)

lemma *inorder_del*:
sorted(*inorder t*) ⟹ *inorder*(*del x t*) = *del_list x* (*inorder t*)
by(*induction x t rule*: *del.induct*)
(*auto simp*: *del_list_simps inorder_baldL inorder_baldR split_minD split*: *prod.splits*)

lemma *inorder_delete*:
sorted(*inorder t*) ⟹ *inorder*(*delete x t*) = *del_list x* (*inorder t*)
by (*auto simp*: *delete_def inorder_del inorder_paint*)

23.2 Structural invariants

lemma *neq_Red*[*simp*]: (*c* ≠ *Red*) = (*c* = *Black*)
by (*cases c*) *auto*

23.2.1 Deletion

lemma *inv_split_min*: ⟦ *split_min t* = (*x*,*t'*); *t* ≠ *Leaf*; *invh t*; *invc t* ⟧
⟹
invh t' ∧
(*color t* = *Red* ⟶ *bheight t'* = *bheight t* ∧ *invc t'*) ∧
(*color t* = *Black* ⟶ *bheight t'* = *bheight t* − *1* ∧ *invc2 t'*)
apply(*induction t arbitrary*: *x t' rule*: *split_min.induct*)
apply(*auto simp*: *inv_baldR inv_baldL invc2I dest*!: *neq_LeafD*
split: *if_splits prod.splits*)
done

An automatic proof. It is quite brittle, e.g. inlining the *let*s in *RBT_Set2.del* breaks it.

lemma *inv_del*: ⟦ *invh t*; *invc t* ⟧ ⟹
invh (*del x t*) ∧

```
  (color t = Red ⟶ bheight (del x t) = bheight t ∧ invc (del x t)) ∧
  (color t = Black ⟶ bheight (del x t) = bheight t − 1 ∧ invc2 (del x t))
apply(induction x t rule: del.induct)
apply(auto simp: inv_baldR inv_baldL invc2I dest!: inv_split_min dest:
neq_LeafD
         split!: prod.splits if_splits)
done
```

A structured proof where one can see what is used in each case.

```
lemma inv_del2: ⟦ invh t; invc t ⟧ ⟹
  invh (del x t) ∧
  (color t = Red ⟶ bheight (del x t) = bheight t ∧ invc (del x t)) ∧
  (color t = Black ⟶ bheight (del x t) = bheight t − 1 ∧ invc2 (del x t))
proof(induction x t rule: del.induct)
 case (1 x)
 then show ?case by simp
next
 case (2 x l a c r)
 note if_split[split del]
 show ?case
 proof cases
  assume x < a
  show ?thesis
  proof cases
   assume l = Leaf thus ?thesis using ‹x < a› 2.prems by(auto)
  next
   assume l: l ≠ Leaf
   show ?thesis
   proof (cases color l)
    assume ∗: color l = Black
    hence bheight l > 0 using l neq_LeafD[of l] by auto
    thus ?thesis using ‹x < a› 2.IH(1) 2.prems inv_baldL[of del x l] ∗
l by(auto)
   next
    assume color l = Red
    thus ?thesis using ‹x < a› 2.prems 2.IH(1) by(auto)
   qed
  qed
 next
  assume ¬ x < a
  show ?thesis
  proof cases
   assume x > a
  show ?thesis using ‹a < x› 2.IH(2) 2.prems neq_LeafD[of r] inv_baldR[of
```

_ *del x r*]
        by(*auto split*: *if_split*)

  next
   assume ¬ $x > a$
   show *?thesis* **using** *2.prems* ‹¬ $x < a$› ‹¬ $x > a$›
    by(*auto simp*: *inv_baldR invc2I dest*!: *inv_split_min dest*: *neq_LeafD split*: *prod.split if_split*)
  qed
 qed
qed

theorem *rbt_delete*: *rbt t* ⟹ *rbt* (*delete x t*)
by (*metis delete_def rbt_def color_paint_Black inv_del invh_paint*)

Overall correctness:

interpretation *S*: *Set_by_Ordered*
where *empty* = *empty* **and** *isin* = *isin* **and** *insert* = *insert* **and** *delete* = *delete*
and *inorder* = *inorder* **and** *inv* = *rbt*
proof (*standard*, *goal_cases*)
 case *1* **show** *?case* **by** (*simp add*: *empty_def*)
next
 case *2* **thus** *?case* **by**(*simp add*: *isin_set_inorder*)
next
 case *3* **thus** *?case* **by**(*simp add*: *inorder_insert*)
next
 case *4* **thus** *?case* **by**(*simp add*: *inorder_delete*)
next
 case *5* **thus** *?case* **by** (*simp add*: *rbt_def empty_def*)
next
 case *6* **thus** *?case* **by** (*simp add*: *rbt_insert*)
next
 case *7* **thus** *?case* **by** (*simp add*: *rbt_delete*)
qed

end

24 Red-Black Tree Implementation of Maps

theory *RBT_Map*
imports
 RBT_Set

Lookup2
begin

fun *upd* :: *′a::linorder* ⇒ *′b* ⇒ (*′a*∗*′b*) *rbt* ⇒ (*′a*∗*′b*) *rbt* **where**
upd x y Leaf = *R Leaf* (*x*,*y*) *Leaf* |
upd x y (*B l* (*a*,*b*) *r*) = (*case cmp x a of*
  LT ⇒ *baliL* (*upd x y l*) (*a*,*b*) *r* |
  GT ⇒ *baliR l* (*a*,*b*) (*upd x y r*) |
  EQ ⇒ *B l* (*x*,*y*) *r*) |
upd x y (*R l* (*a*,*b*) *r*) = (*case cmp x a of*
  LT ⇒ *R* (*upd x y l*) (*a*,*b*) *r* |
  GT ⇒ *R l* (*a*,*b*) (*upd x y r*) |
  EQ ⇒ *R l* (*x*,*y*) *r*)

definition *update* :: *′a::linorder* ⇒ *′b* ⇒ (*′a*∗*′b*) *rbt* ⇒ (*′a*∗*′b*) *rbt* **where**
update x y t = *paint Black* (*upd x y t*)

fun *del* :: *′a::linorder* ⇒ (*′a*∗*′b*)*rbt* ⇒ (*′a*∗*′b*)*rbt* **where**
del x Leaf = *Leaf* |
del x (*Node l* (*ab*, _) *r*) = (*case cmp x* (*fst ab*) *of*
    LT ⇒ *if l* ≠ *Leaf* ∧ *color l* = *Black*
        then baldL (*del x l*) *ab r else R* (*del x l*) *ab r* |
    GT ⇒ *if r* ≠ *Leaf*∧ *color r* = *Black*
        then baldR l ab (*del x r*) *else R l ab* (*del x r*) |
  EQ ⇒ *join l r*)

definition *delete* :: *′a::linorder* ⇒ (*′a*∗*′b*) *rbt* ⇒ (*′a*∗*′b*) *rbt* **where**
delete x t = *paint Black* (*del x t*)

24.1 Functional Correctness Proofs

lemma *inorder_upd*:
  sorted1(*inorder t*) ⟹ *inorder*(*upd x y t*) = *upd_list x y* (*inorder t*)
by(*induction x y t rule*: *upd.induct*)
  (*auto simp*: *upd_list_simps inorder_baliL inorder_baliR*)

lemma *inorder_update*:
  sorted1(*inorder t*) ⟹ *inorder*(*update x y t*) = *upd_list x y* (*inorder t*)
by(*simp add*: *update_def inorder_upd inorder_paint*)

lemma *del_list_id*: ∀ *ab*∈*set ps*. *y* < *fst ab* ⟹ *x* ≤ *y* ⟹ *del_list x ps* = *ps*
by(*rule del_list_idem*) *auto*

lemma *inorder_del*:
sorted1(*inorder t*) $\Longrightarrow$ *inorder*(*del x t*) = *del_list x* (*inorder t*)
by(*induction x t rule*: *del.induct*)
(*auto simp*: *del_list_simps del_list_id inorder_join inorder_baldL inorder_baldR*)

lemma *inorder_delete*:
sorted1(*inorder t*) $\Longrightarrow$ *inorder*(*delete x t*) = *del_list x* (*inorder t*)
by(*simp add*: *delete_def inorder_del inorder_paint*)

24.2 Structural invariants

24.2.1 Update

lemma *invc_upd*: **assumes** *invc t*
shows *color t* = *Black* $\Longrightarrow$ *invc* (*upd x y t*) *invc2* (*upd x y t*)
using *assms*
by (*induct x y t rule*: *upd.induct*) (*auto simp*: *invc_baliL invc_baliR invc2I*)

lemma *invh_upd*: **assumes** *invh t*
shows *invh* (*upd x y t*) *bheight* (*upd x y t*) = *bheight t*
using *assms*
by(*induct x y t rule*: *upd.induct*)
(*auto simp*: *invh_baliL invh_baliR bheight_baliL bheight_baliR*)

theorem *rbt_update*: *rbt t* $\Longrightarrow$ *rbt* (*update x y t*)
by (*simp add*: *invc_upd*(*2*) *invh_upd*(*1*) *color_paint_Black invh_paint rbt_def update_def*)

24.2.2 Deletion

lemma *del_invc_invh*: *invh t* $\Longrightarrow$ *invc t* $\Longrightarrow$ *invh* (*del x t*) $\wedge$
(*color t* = *Red* $\wedge$ *bheight* (*del x t*) = *bheight t* $\wedge$ *invc* (*del x t*) $\vee$
color t = *Black* $\wedge$ *bheight* (*del x t*) = *bheight t* − *1* $\wedge$ *invc2* (*del x t*))
proof (*induct x t rule*: *del.induct*)
case (*2 x _ ab c*)
have *x* = *fst ab* $\vee$ *x* < *fst ab* $\vee$ *x* > *fst ab* **by** *auto*
thus *?case* **proof** (*elim disjE*)
assume *x* = *fst ab*
with *2* **show** *?thesis*
by (*cases c*) (*simp_all add*: *invh_join invc_join*)
next
assume *x* < *fst ab*
with *2* **show** *?thesis*

```
    by(cases c)
    (auto simp: invh_baldL_invc invc_baldL invc2_baldL dest: neq_LeafD)
  next
   assume fst ab < x
   with 2 show ?thesis
    by(cases c)
    (auto simp: invh_baldR_invc invc_baldR invc2_baldR dest: neq_LeafD)
  qed
qed auto

theorem rbt_delete: rbt t ⟹ rbt (delete k t)
by (metis delete_def rbt_def color_paint_Black del_invc_invh invc2I invh_paint)

interpretation M: Map_by_Ordered
where empty = empty and lookup = lookup and update = update and
delete = delete
and inorder = inorder and inv = rbt
proof (standard, goal_cases)
  case 1 show ?case by (simp add: empty_def)
next
  case 2 thus ?case by(simp add: lookup_map_of)
next
  case 3 thus ?case by(simp add: inorder_update)
next
  case 4 thus ?case by(simp add: inorder_delete)
next
  case 5 thus ?case by (simp add: rbt_def empty_def)
next
  case 6 thus ?case by (simp add: rbt_update)
next
  case 7 thus ?case by (simp add: rbt_delete)
qed

end
```

25 2-3 Trees

```
theory Tree23
imports Main
begin

class height =
fixes height :: 'a ⇒ nat
```

datatype *'a tree23* =
  Leaf (⟨⟩) |
  Node2 'a tree23 'a 'a tree23 (⟨_, _, _⟩) |
  Node3 'a tree23 'a 'a tree23 'a 'a tree23 (⟨_, _, _, _, _⟩)

fun *inorder* :: *'a tree23* ⇒ *'a list* **where**
inorder Leaf = [] |
inorder(*Node2 l a r*) = *inorder l* @ *a* # *inorder r* |
inorder(*Node3 l a m b r*) = *inorder l* @ *a* # *inorder m* @ *b* # *inorder r*

instantiation *tree23* :: (*type*)*height*
begin

fun *height_tree23* :: *'a tree23* ⇒ *nat* **where**
height Leaf = *0* |
height (*Node2 l _ r*) = *Suc*(*max* (*height l*) (*height r*)) |
height (*Node3 l _ m _ r*) = *Suc*(*max* (*height l*) (*max* (*height m*) (*height r*)))

instance ..

end

Completeness:

fun *complete* :: *'a tree23* ⇒ *bool* **where**
complete Leaf = *True* |
complete (*Node2 l _ r*) = (*height l* = *height r* ∧ *complete l* & *complete r*) |
complete (*Node3 l _ m _ r*) =
  (*height l* = *height m* & *height m* = *height r* & *complete l* & *complete m* & *complete r*)

lemma *ht_sz_if_complete*: *complete t* ⟹ *2* ^ *height t* ≤ *size t* + *1*
by (*induction t*) *auto*

end

26 2-3 Tree Implementation of Sets

theory *Tree23_Set*
imports
  Tree23
  Cmp

```
  Set_Specs
begin

declare sorted_wrt.simps(2)[simp del]

definition empty :: 'a tree23 where
empty = Leaf

fun isin :: 'a::linorder tree23 ⇒ 'a ⇒ bool where
isin Leaf x = False |
isin (Node2 l a r) x =
  (case cmp x a of
     LT ⇒ isin l x |
     EQ ⇒ True |
     GT ⇒ isin r x) |
isin (Node3 l a m b r) x =
  (case cmp x a of
     LT ⇒ isin l x |
     EQ ⇒ True |
     GT ⇒
       (case cmp x b of
          LT ⇒ isin m x |
          EQ ⇒ True |
          GT ⇒ isin r x))

datatype 'a upI = TI 'a tree23 | OF 'a tree23 'a 'a tree23

fun treeI :: 'a upI ⇒ 'a tree23 where
treeI (TI t) = t |
treeI (OF l a r) = Node2 l a r

fun ins :: 'a::linorder ⇒ 'a tree23 ⇒ 'a upI where
ins x Leaf = OF Leaf x Leaf |
ins x (Node2 l a r) =
   (case cmp x a of
      LT ⇒
        (case ins x l of
           TI l' => TI (Node2 l' a r) |
           OF l1 b l2 => TI (Node3 l1 b l2 a r)) |
      EQ ⇒ TI (Node2 l a r) |
      GT ⇒
        (case ins x r of
           TI r' => TI (Node2 l a r') |
           OF r1 b r2 => TI (Node3 l a r1 b r2))) |
```

```
ins x (Node3 l a m b r) =
  (case cmp x a of
     LT ⇒
       (case ins x l of
          TI l' => TI (Node3 l' a m b r) |
          OF l1 c l2 => OF (Node2 l1 c l2) a (Node2 m b r)) |
     EQ ⇒ TI (Node3 l a m b r) |
     GT ⇒
       (case cmp x b of
          GT ⇒
            (case ins x r of
               TI r' => TI (Node3 l a m b r') |
               OF r1 c r2 => OF (Node2 l a m) b (Node2 r1 c r2)) |
          EQ ⇒ TI (Node3 l a m b r) |
          LT ⇒
            (case ins x m of
               TI m' => TI (Node3 l a m' b r) |
               OF m1 c m2 => OF (Node2 l a m1) c (Node2 m2 b r))))
```

hide_const *insert*

definition *insert* :: *'a::linorder ⇒ 'a tree23 ⇒ 'a tree23* **where**
insert x t = treeI(ins x t)

datatype *'a upD = TD 'a tree23 | UF 'a tree23*

fun *treeD* :: *'a upD ⇒ 'a tree23* **where**
treeD (TD t) = t |
treeD (UF t) = t

fun *node21* :: *'a upD ⇒ 'a ⇒ 'a tree23 ⇒ 'a upD* **where**
node21 (TD t1) a t2 = TD(Node2 t1 a t2) |
node21 (UF t1) a (Node2 t2 b t3) = UF(Node3 t1 a t2 b t3) |
node21 (UF t1) a (Node3 t2 b t3 c t4) = TD(Node2 (Node2 t1 a t2) b (Node2 t3 c t4))

fun *node22* :: *'a tree23 ⇒ 'a ⇒ 'a upD ⇒ 'a upD* **where**
node22 t1 a (TD t2) = TD(Node2 t1 a t2) |
node22 (Node2 t1 b t2) a (UF t3) = UF(Node3 t1 b t2 a t3) |
node22 (Node3 t1 b t2 c t3) a (UF t4) = TD(Node2 (Node2 t1 b t2) c (Node2 t3 a t4))

fun *node31* :: *'a upD* ⇒ *'a* ⇒ *'a tree23* ⇒ *'a* ⇒ *'a tree23* ⇒ *'a upD* **where**
node31 (TD t1) a t2 b t3 = TD(Node3 t1 a t2 b t3) |
node31 (UF t1) a (Node2 t2 b t3) c t4 = TD(Node2 (Node3 t1 a t2 b t3) c t4) |
node31 (UF t1) a (Node3 t2 b t3 c t4) d t5 = TD(Node3 (Node2 t1 a t2) b (Node2 t3 c t4) d t5)

fun *node32* :: *'a tree23* ⇒ *'a* ⇒ *'a upD* ⇒ *'a* ⇒ *'a tree23* ⇒ *'a upD* **where**
node32 t1 a (TD t2) b t3 = TD(Node3 t1 a t2 b t3) |
node32 t1 a (UF t2) b (Node2 t3 c t4) = TD(Node2 t1 a (Node3 t2 b t3 c t4)) |
node32 t1 a (UF t2) b (Node3 t3 c t4 d t5) = TD(Node3 t1 a (Node2 t2 b t3) c (Node2 t4 d t5))

fun *node33* :: *'a tree23* ⇒ *'a* ⇒ *'a tree23* ⇒ *'a* ⇒ *'a upD* ⇒ *'a upD* **where**
node33 t1 a t2 b (TD t3) = TD(Node3 t1 a t2 b t3) |
node33 t1 a (Node2 t2 b t3) c (UF t4) = TD(Node2 t1 a (Node3 t2 b t3 c t4)) |
node33 t1 a (Node3 t2 b t3 c t4) d (UF t5) = TD(Node3 t1 a (Node2 t2 b t3) c (Node2 t4 d t5))

fun *split_min* :: *'a tree23* ⇒ *'a* ∗ *'a upD* **where**
split_min (Node2 Leaf a Leaf) = (a, UF Leaf) |
split_min (Node3 Leaf a Leaf b Leaf) = (a, TD(Node2 Leaf b Leaf)) |
split_min (Node2 l a r) = (let (x,l') = split_min l in (x, node21 l' a r)) |
split_min (Node3 l a m b r) = (let (x,l') = split_min l in (x, node31 l' a m b r))

In the base cases of *split_min* and *del* it is enough to check if one subtree is a *Leaf*, in which case completeness implies that so are the others. Exercise.

fun *del* :: *'a::linorder* ⇒ *'a tree23* ⇒ *'a upD* **where**
del x Leaf = TD Leaf |
del x (Node2 Leaf a Leaf) =
  (if x = a then UF Leaf else TD(Node2 Leaf a Leaf)) |
del x (Node3 Leaf a Leaf b Leaf) =
  TD(if x = a then Node2 Leaf b Leaf else
    if x = b then Node2 Leaf a Leaf
    else Node3 Leaf a Leaf b Leaf) |
del x (Node2 l a r) =
  (case cmp x a of
    LT ⇒ node21 (del x l) a r |
    GT ⇒ node22 l a (del x r) |
    EQ ⇒ let (a',r') = split_min r in node22 l a' r') |
del x (Node3 l a m b r) =

(*case cmp x a of*
  LT ⇒ *node31* (*del x l*) *a m b r* |
  EQ ⇒ *let* (*a′*,*m′*) = *split_min m in node32 l a′ m′ b r* |
  GT ⇒
   (*case cmp x b of*
     LT ⇒ *node32 l a* (*del x m*) *b r* |
     EQ ⇒ *let* (*b′*,*r′*) = *split_min r in node33 l a m b′ r′* |
     GT ⇒ *node33 l a m b* (*del x r*)))

definition *delete* :: *′a*::*linorder* ⇒ *′a tree23* ⇒ *′a tree23* **where**
delete x t = *treeD*(*del x t*)

26.1 Functional Correctness

26.1.1 Proofs for isin

lemma *isin_set*: *sorted*(*inorder t*) ⟹ *isin t x* = (*x* ∈ *set* (*inorder t*))
by (*induction t*) (*auto simp*: *isin_simps*)

26.1.2 Proofs for insert

lemma *inorder_ins*:
  sorted(*inorder t*) ⟹ *inorder*(*treeI*(*ins x t*)) = *ins_list x* (*inorder t*)
by(*induction t*) (*auto simp*: *ins_list_simps split*: *upI.splits*)

lemma *inorder_insert*:
  sorted(*inorder t*) ⟹ *inorder*(*insert a t*) = *ins_list a* (*inorder t*)
by(*simp add*: *insert_def inorder_ins*)

26.1.3 Proofs for delete

lemma *inorder_node21*: *height r* > *0* ⟹
  inorder (*treeD* (*node21 l′ a r*)) = *inorder* (*treeD l′*) @ *a* # *inorder r*
by(*induct l′ a r rule*: *node21.induct*) *auto*

lemma *inorder_node22*: *height l* > *0* ⟹
  inorder (*treeD* (*node22 l a r′*)) = *inorder l* @ *a* # *inorder* (*treeD r′*)
by(*induct l a r′ rule*: *node22.induct*) *auto*

lemma *inorder_node31*: *height m* > *0* ⟹
  inorder (*treeD* (*node31 l′ a m b r*)) = *inorder* (*treeD l′*) @ *a* # *inorder m*
@ *b* # *inorder r*
by(*induct l′ a m b r rule*: *node31.induct*) *auto*

lemma *inorder_node32*: *height r* > *0* ⟹

inorder (*treeD* (*node32 l a m′ b r*)) = *inorder l* @ *a* # *inorder* (*treeD m′*) @ *b* # *inorder r*
by(*induct l a m′ b r rule*: *node32.induct*) *auto*

lemma *inorder_node33*: *height m* > *0* ⟹
inorder (*treeD* (*node33 l a m b r′*)) = *inorder l* @ *a* # *inorder m* @ *b* # *inorder* (*treeD r′*)
by(*induct l a m b r′ rule*: *node33.induct*) *auto*

lemmas *inorder_nodes* = *inorder_node21 inorder_node22*
inorder_node31 inorder_node32 inorder_node33

lemma *split_minD*:
split_min t = (*x*,*t′*) ⟹ *complete t* ⟹ *height t* > *0* ⟹
x # *inorder*(*treeD t′*) = *inorder t*
by(*induction t arbitrary*: *t′ rule*: *split_min.induct*)
(*auto simp*: *inorder_nodes split*: *prod.splits*)

lemma *inorder_del*: ⟦ *complete t* ; *sorted*(*inorder t*) ⟧ ⟹
inorder(*treeD* (*del x t*)) = *del_list x* (*inorder t*)
by(*induction t rule*: *del.induct*)
(*auto simp*: *del_list_simps inorder_nodes split_minD split*!: *if_split prod.splits*)

lemma *inorder_delete*: ⟦ *complete t* ; *sorted*(*inorder t*) ⟧ ⟹
inorder(*delete x t*) = *del_list x* (*inorder t*)
by(*simp add*: *delete_def inorder_del*)

26.2 Completeness

26.2.1 Proofs for insert

First a standard proof that *ins* preserves *complete*.

fun *hI* :: ′*a upI* ⇒ *nat* **where**
hI (*TI t*) = *height t* |
hI (*OF l a r*) = *height l*

lemma *complete_ins*: *complete t* ⟹ *complete* (*treeI*(*ins a t*)) ∧ *hI*(*ins a t*) = *height t*
by (*induct t*) (*auto split*!: *if_split upI.split*)

Now an alternative proof (by Brian Huffman) that runs faster because two properties (completeness and height) are combined in one predicate.

inductive *full* :: *nat* ⇒ ′*a tree23* ⇒ *bool* **where**
full 0 Leaf |

$[\![$*full n l*; *full n r*$]\!]$ $\Longrightarrow$ *full* (*Suc n*) (*Node2 l p r*) |
$[\![$*full n l*; *full n m*; *full n r*$]\!]$ $\Longrightarrow$ *full* (*Suc n*) (*Node3 l p m q r*)

inductive_cases *full_elims*:
  full n Leaf
  full n (*Node2 l p r*)
  full n (*Node3 l p m q r*)

inductive_cases *full_0_elim*: *full 0 t*
inductive_cases *full_Suc_elim*: *full* (*Suc n*) *t*

lemma *full_0_iff* [*simp*]: *full 0 t* $\longleftrightarrow$ *t* = *Leaf*
  by (*auto elim*: *full_0_elim intro*: *full.intros*)

lemma *full_Leaf_iff* [*simp*]: *full n Leaf* $\longleftrightarrow$ *n* = *0*
  by (*auto elim*: *full_elims intro*: *full.intros*)

lemma *full_Suc_Node2_iff* [*simp*]:
  full (*Suc n*) (*Node2 l p r*) $\longleftrightarrow$ *full n l* $\wedge$ *full n r*
  by (*auto elim*: *full_elims intro*: *full.intros*)

lemma *full_Suc_Node3_iff* [*simp*]:
  full (*Suc n*) (*Node3 l p m q r*) $\longleftrightarrow$ *full n l* $\wedge$ *full n m* $\wedge$ *full n r*
  by (*auto elim*: *full_elims intro*: *full.intros*)

lemma *full_imp_height*: *full n t* $\Longrightarrow$ *height t* = *n*
  by (*induct set*: *full*, *simp_all*)

lemma *full_imp_complete*: *full n t* $\Longrightarrow$ *complete t*
  by (*induct set*: *full*, *auto dest*: *full_imp_height*)

lemma *complete_imp_full*: *complete t* $\Longrightarrow$ *full* (*height t*) *t*
  by (*induct t*, *simp_all*)

lemma *complete_iff_full*: *complete t* $\longleftrightarrow$ ($\exists$ *n*. *full n t*)
  by (*auto elim*!: *complete_imp_full full_imp_complete*)

The *insert* function either preserves the height of the tree, or increases it by one. The constructor returned by the *insert* function determines which: A return value of the form *TI t* indicates that the height will be the same. A value of the form *OF l p r* indicates an increase in height.

fun $full_i$ :: *nat* $\Rightarrow$ *'a upI* $\Rightarrow$ *bool* **where**
$full_i$ *n* (*TI t*) $\longleftrightarrow$ *full n t* |
$full_i$ *n* (*OF l p r*) $\longleftrightarrow$ *full n l* $\wedge$ *full n r*

lemma *full$_i$_ins*: *full n t* $\Longrightarrow$ *full$_i$ n* (*ins a t*)
by (*induct rule*: *full.induct*) (*auto split*: *upI.split*)

The *insert* operation preserves completeance.

lemma *complete_insert*: *complete t* $\Longrightarrow$ *complete* (*insert a t*)
unfolding *complete_iff_full insert_def*
apply (*erule exE*)
apply (*drule full$_i$_ins* [*of* _ _ *a*])
apply (*cases ins a t*)
apply (*auto intro*: *full.intros*)
done

26.3 Proofs for delete

fun *hD* :: *'a upD* ⇒ *nat* **where**
hD (*TD t*) = *height t* |
hD (*UF t*) = *height t* + *1*

lemma *complete_treeD_node21*:
⟦*complete r*; *complete* (*treeD l'*); *height r* = *hD l'* ⟧ $\Longrightarrow$ *complete* (*treeD* (*node21 l' a r*))
by(*induct l' a r rule*: *node21.induct*) *auto*

lemma *complete_treeD_node22*:
⟦*complete*(*treeD r'*); *complete l*; *hD r'* = *height l* ⟧ $\Longrightarrow$ *complete* (*treeD* (*node22 l a r'*))
by(*induct l a r' rule*: *node22.induct*) *auto*

lemma *complete_treeD_node31*:
⟦ *complete* (*treeD l'*); *complete m*; *complete r*; *hD l'* = *height r*; *height m* = *height r* ⟧
$\Longrightarrow$ *complete* (*treeD* (*node31 l' a m b r*))
by(*induct l' a m b r rule*: *node31.induct*) *auto*

lemma *complete_treeD_node32*:
⟦ *complete l*; *complete* (*treeD m'*); *complete r*; *height l* = *height r*; *hD m'* = *height r* ⟧
$\Longrightarrow$ *complete* (*treeD* (*node32 l a m' b r*))
by(*induct l a m' b r rule*: *node32.induct*) *auto*

lemma *complete_treeD_node33*:
⟦ *complete l*; *complete m*; *complete*(*treeD r'*); *height l* = *hD r'*; *height m* = *hD r'* ⟧

$\Longrightarrow$ *complete* (*treeD* (*node33 l a m b r'*))
by(*induct l a m b r' rule*: *node33.induct*) *auto*

lemmas *completes* = *complete_treeD_node21 complete_treeD_node22 complete_treeD_node31 complete_treeD_node32 complete_treeD_node33*

lemma *height'_node21*:
height r > *0* $\Longrightarrow$ *hD*(*node21 l' a r*) = *max* (*hD l'*) (*height r*) + *1*
by(*induct l' a r rule*: *node21.induct*)(*simp_all*)

lemma *height'_node22*:
height l > *0* $\Longrightarrow$ *hD*(*node22 l a r'*) = *max* (*height l*) (*hD r'*) + *1*
by(*induct l a r' rule*: *node22.induct*)(*simp_all*)

lemma *height'_node31*:
height m > *0* $\Longrightarrow$ *hD*(*node31 l a m b r*) =
max (*hD l*) (*max* (*height m*) (*height r*)) + *1*
by(*induct l a m b r rule*: *node31.induct*)(*simp_all add*: *max_def*)

lemma *height'_node32*:
height r > *0* $\Longrightarrow$ *hD*(*node32 l a m b r*) =
max (*height l*) (*max* (*hD m*) (*height r*)) + *1*
by(*induct l a m b r rule*: *node32.induct*)(*simp_all add*: *max_def*)

lemma *height'_node33*:
height m > *0* $\Longrightarrow$ *hD*(*node33 l a m b r*) =
max (*height l*) (*max* (*height m*) (*hD r*)) + *1*
by(*induct l a m b r rule*: *node33.induct*)(*simp_all add*: *max_def*)

lemmas *heights* = *height'_node21 height'_node22 height'_node31 height'_node32 height'_node33*

lemma *height_split_min*:
split_min t = (*x*, *t'*) $\Longrightarrow$ *height t* > *0* $\Longrightarrow$ *complete t* $\Longrightarrow$ *hD t'* = *height t*
by(*induct t arbitrary*: *x t' rule*: *split_min.induct*)
(*auto simp*: *heights split*: *prod.splits*)

lemma *height_del*: *complete t* $\Longrightarrow$ *hD*(*del x t*) = *height t*
by(*induction x t rule*: *del.induct*)
(*auto simp*: *heights max_def height_split_min split*: *prod.splits*)

lemma *complete_split_min*:
⟦ *split_min t* = (*x*, *t'*); *complete t*; *height t* > *0* ⟧ $\Longrightarrow$ *complete* (*treeD t'*)

by(*induct t arbitrary*: *x t′ rule*: *split_min.induct*)
 (*auto simp*: *heights height_split_min completes split*: *prod.splits*)

lemma *complete_treeD_del*: *complete t* ⟹ *complete*(*treeD*(*del x t*))
by(*induction x t rule*: *del.induct*)
 (*auto simp*: *completes complete_split_min height_del height_split_min split*: *prod.splits*)

corollary *complete_delete*: *complete t* ⟹ *complete*(*delete x t*)
by(*simp add*: *delete_def complete_treeD_del*)

26.4 Overall Correctness

interpretation *S*: *Set_by_Ordered*
where *empty* = *empty* **and** *isin* = *isin* **and** *insert* = *insert* **and** *delete* = *delete*
and *inorder* = *inorder* **and** *inv* = *complete*
proof (*standard*, *goal_cases*)
 case *2* **thus** *?case* **by**(*simp add*: *isin_set*)
next
 case *3* **thus** *?case* **by**(*simp add*: *inorder_insert*)
next
 case *4* **thus** *?case* **by**(*simp add*: *inorder_delete*)
next
 case *6* **thus** *?case* **by**(*simp add*: *complete_insert*)
next
 case *7* **thus** *?case* **by**(*simp add*: *complete_delete*)
qed (*simp add*: *empty_def*)+

end

27 2-3 Tree Implementation of Maps

theory *Tree23_Map*
imports
 Tree23_Set
 Map_Specs
begin

fun *lookup* :: (*′a*::*linorder* ∗ *′b*) *tree23* ⇒ *′a* ⇒ *′b option* **where**
lookup Leaf x = *None* |
lookup (*Node2 l* (*a*,*b*) *r*) *x* = (*case cmp x a of*
 LT ⇒ *lookup l x* |
 GT ⇒ *lookup r x* |

```
  EQ ⇒ Some b) |
lookup (Node3 l (a1,b1) m (a2,b2) r) x = (case cmp x a1 of
  LT ⇒ lookup l x |
  EQ ⇒ Some b1 |
  GT ⇒ (case cmp x a2 of
          LT ⇒ lookup m x |
          EQ ⇒ Some b2 |
          GT ⇒ lookup r x))

fun upd :: 'a::linorder ⇒ 'b ⇒ ('a∗'b) tree23 ⇒ ('a∗'b) upI where
upd x y Leaf = OF Leaf (x,y) Leaf |
upd x y (Node2 l ab r) = (case cmp x (fst ab) of
   LT ⇒ (case upd x y l of
          TI l' => TI (Node2 l' ab r)
        | OF l1 ab' l2 => TI (Node3 l1 ab' l2 ab r)) |
   EQ ⇒ TI (Node2 l (x,y) r) |
   GT ⇒ (case upd x y r of
          TI r' => TI (Node2 l ab r')
        | OF r1 ab' r2 => TI (Node3 l ab r1 ab' r2))) |
upd x y (Node3 l ab1 m ab2 r) = (case cmp x (fst ab1) of
   LT ⇒ (case upd x y l of
          TI l' => TI (Node3 l' ab1 m ab2 r)
        | OF l1 ab' l2 => OF (Node2 l1 ab' l2) ab1 (Node2 m ab2 r)) |
   EQ ⇒ TI (Node3 l (x,y) m ab2 r) |
   GT ⇒ (case cmp x (fst ab2) of
          LT ⇒ (case upd x y m of
                 TI m' => TI (Node3 l ab1 m' ab2 r)
               | OF m1 ab' m2 => OF (Node2 l ab1 m1) ab' (Node2 m2
ab2 r)) |
          EQ ⇒ TI (Node3 l ab1 m (x,y) r) |
          GT ⇒ (case upd x y r of
                 TI r' => TI (Node3 l ab1 m ab2 r')
               | OF r1 ab' r2 => OF (Node2 l ab1 m) ab2 (Node2 r1 ab'
r2))))

definition update :: 'a::linorder ⇒ 'b ⇒ ('a∗'b) tree23 ⇒ ('a∗'b) tree23
where
update a b t = treeI(upd a b t)

fun del :: 'a::linorder ⇒ ('a∗'b) tree23 ⇒ ('a∗'b) upD where
del x Leaf = TD Leaf |
del x (Node2 Leaf ab1 Leaf) = (if x=fst ab1 then UF Leaf else TD(Node2
Leaf ab1 Leaf)) |
del x (Node3 Leaf ab1 Leaf ab2 Leaf) = TD(if x=fst ab1 then Node2 Leaf
```

ab2 Leaf
  else if x=fst ab2 then Node2 Leaf ab1 Leaf else Node3 Leaf ab1 Leaf ab2 Leaf) |
del x (*Node2 l ab1 r*) = (*case cmp x* (*fst ab1*) *of*
  LT ⇒ *node21* (*del x l*) *ab1 r* |
  GT ⇒ *node22 l ab1* (*del x r*) |
  EQ ⇒ *let* (*ab1'*,*t*) = *split_min r in node22 l ab1' t*) |
del x (*Node3 l ab1 m ab2 r*) = (*case cmp x* (*fst ab1*) *of*
  LT ⇒ *node31* (*del x l*) *ab1 m ab2 r* |
  EQ ⇒ *let* (*ab1'*,*m'*) = *split_min m in node32 l ab1' m' ab2 r* |
  GT ⇒ (*case cmp x* (*fst ab2*) *of*
      LT ⇒ *node32 l ab1* (*del x m*) *ab2 r* |
      EQ ⇒ *let* (*ab2'*,*r'*) = *split_min r in node33 l ab1 m ab2' r'* |
      GT ⇒ *node33 l ab1 m ab2* (*del x r*)))

definition *delete* :: *'a::linorder* ⇒ (*'a***'b*) *tree23* ⇒ (*'a***'b*) *tree23* **where**
delete x t = *treeD*(*del x t*)

27.1 Functional Correctness

lemma *lookup_map_of*:
  sorted1(*inorder t*) ⟹ *lookup t x* = *map_of* (*inorder t*) *x*
by (*induction t*) (*auto simp*: *map_of_simps split*: *option.split*)

lemma *inorder_upd*:
  sorted1(*inorder t*) ⟹ *inorder*(*treeI*(*upd x y t*)) = *upd_list x y* (*inorder t*)
by(*induction t*) (*auto simp*: *upd_list_simps split*: *upI.splits*)

corollary *inorder_update*:
  sorted1(*inorder t*) ⟹ *inorder*(*update x y t*) = *upd_list x y* (*inorder t*)
by(*simp add*: *update_def inorder_upd*)

lemma *inorder_del*: ⟦ *complete t* ; *sorted1*(*inorder t*) ⟧ ⟹
  inorder(*treeD* (*del x t*)) = *del_list x* (*inorder t*)
by(*induction t rule*: *del.induct*)
 (*auto simp*: *del_list_simps inorder_nodes split_minD split*!: *if_split prod.splits*)

corollary *inorder_delete*: ⟦ *complete t* ; *sorted1*(*inorder t*) ⟧ ⟹
  inorder(*delete x t*) = *del_list x* (*inorder t*)
by(*simp add*: *delete_def inorder_del*)

27.2 Balancedness

lemma *complete_upd*: *complete t* $\Longrightarrow$ *complete* (*treeI*(*upd x y t*)) $\wedge$ *hI*(*upd x y t*) = *height t*
by (*induct t*) (*auto split*!: *if_split upI.split*)

corollary *complete_update*: *complete t* $\Longrightarrow$ *complete* (*update x y t*)
by (*simp add*: *update_def complete_upd*)

lemma *height_del*: *complete t* $\Longrightarrow$ *hD*(*del x t*) = *height t*
by(*induction x t rule*: *del.induct*)
(*auto simp add*: *heights max_def height_split_min split*: *prod.split*)

lemma *complete_treeD_del*: *complete t* $\Longrightarrow$ *complete*(*treeD*(*del x t*))
by(*induction x t rule*: *del.induct*)
(*auto simp*: *completes complete_split_min height_del height_split_min split*: *prod.split*)

corollary *complete_delete*: *complete t* $\Longrightarrow$ *complete*(*delete x t*)
by(*simp add*: *delete_def complete_treeD_del*)

27.3 Overall Correctness

interpretation *M*: *Map_by_Ordered*
where *empty* = *empty* **and** *lookup* = *lookup* **and** *update* = *update* **and** *delete* = *delete*
and *inorder* = *inorder* **and** *inv* = *complete*
proof (*standard*, *goal_cases*)
case *1* **thus** *?case* **by**(*simp add*: *empty_def*)
next
case *2* **thus** *?case* **by**(*simp add*: *lookup_map_of*)
next
case *3* **thus** *?case* **by**(*simp add*: *inorder_update*)
next
case *4* **thus** *?case* **by**(*simp add*: *inorder_delete*)
next
case *5* **thus** *?case* **by**(*simp add*: *empty_def*)
next
case *6* **thus** *?case* **by**(*simp add*: *complete_update*)
next
case *7* **thus** *?case* **by**(*simp add*: *complete_delete*)
qed

end

28 2-3 Tree from List

theory *Tree23_of_List*
imports *Tree23*
begin

Linear-time bottom up conversion of a list of items into a complete 2-3 tree whose inorder traversal yields the list of items.

28.1 Code

Nonempty lists of 2-3 trees alternating with items, starting and ending with a 2-3 tree:

datatype *'a tree23s = T 'a tree23 | TTs 'a tree23 'a 'a tree23s*

abbreviation *not_T ts* == (∀ *t. ts* ≠ *T t*)

fun *len* :: *'a tree23s* ⇒ *nat* **where**
len (*T* _) = *1* |
len (*TTs* _ _ *ts*) = *len ts* + *1*

fun *trees* :: *'a tree23s* ⇒ *'a tree23 set* **where**
trees (*T t*) = {*t*} |
trees (*TTs t a ts*) = {*t*} ∪ *trees ts*

Join pairs of adjacent trees:

fun *join_adj* :: *'a tree23s* ⇒ *'a tree23s* **where**
join_adj (*TTs t1 a* (*T t2*)) = *T*(*Node2 t1 a t2*) |
join_adj (*TTs t1 a* (*TTs t2 b* (*T t3*))) = *T*(*Node3 t1 a t2 b t3*) |
join_adj (*TTs t1 a* (*TTs t2 b ts*)) = *TTs* (*Node2 t1 a t2*) *b* (*join_adj ts*)

Towards termination of *join_all*:

lemma *len_ge2*:
  not_T ts ⟹ *len ts* ≥ *2*
by(*cases ts rule*: *join_adj.cases*) *auto*

lemma [*measure_function*]: *is_measure len*
by(*rule is_measure_trivial*)

lemma *len_join_adj_div2*:
  not_T ts ⟹ *len*(*join_adj ts*) ≤ *len ts div 2*
by(*induction ts rule*: *join_adj.induct*) *auto*

lemma *len_join_adj1*: *not_T ts* ⟹ *len*(*join_adj ts*) < *len ts*
using *len_join_adj_div2*[*of ts*] *len_ge2*[*of ts*] **by** *simp*

corollary *len_join_adj2*[*termination_simp*]: *len*(*join_adj* (*TTs t a ts*)) ≤ *len ts*
using *len_join_adj1*[*of TTs t a ts*] **by** *simp*

fun *join_all* :: *'a tree23s* ⇒ *'a tree23* **where**
join_all (*T t*) = *t* |
join_all ts = *join_all* (*join_adj ts*)

fun *leaves* :: *'a list* ⇒ *'a tree23s* **where**
leaves [] = *T Leaf* |
leaves (*a* # *as*) = *TTs Leaf a* (*leaves as*)

definition *tree23_of_list* :: *'a list* ⇒ *'a tree23* **where**
tree23_of_list as = *join_all*(*leaves as*)

28.2 Functional correctness

28.2.1 *inorder*:

fun *inorder2* :: *'a tree23s* ⇒ *'a list* **where**
inorder2 (*T t*) = *inorder t* |
inorder2 (*TTs t a ts*) = *inorder t* @ *a* # *inorder2 ts*

lemma *inorder2_join_adj*: *not_T ts* ⟹ *inorder2*(*join_adj ts*) = *inorder2 ts*
by (*induction ts rule*: *join_adj.induct*) *auto*

lemma *inorder_join_all*: *inorder* (*join_all ts*) = *inorder2 ts*
proof (*induction ts rule*: *join_all.induct*)
  case *1* **thus** *?case* **by** *simp*
next
  case (*2 t a ts*)
  thus *?case* **using** *inorder2_join_adj*[*of TTs t a ts*]
    by (*simp add*: *le_imp_less_Suc*)
qed

lemma *inorder2_leaves*: *inorder2*(*leaves as*) = *as*
by(*induction as*) *auto*

lemma *inorder*: *inorder*(*tree23_of_list as*) = *as*

by(*simp add*: *tree23_of_list_def inorder_join_all inorder2_leaves*)

28.2.2 Completeness:

lemma *complete_join_adj*:
 ∀ *t* ∈ *trees ts*. *complete t* ∧ *height t* = *n* ⟹ *not_T ts* ⟹
  ∀ *t* ∈ *trees* (*join_adj ts*). *complete t* ∧ *height t* = *Suc n*
by (*induction ts rule*: *join_adj.induct*) *auto*

lemma *complete_join_all*:
∀ *t* ∈ *trees ts*. *complete t* ∧ *height t* = *n* ⟹ *complete* (*join_all ts*)
proof (*induction ts arbitrary*: *n rule*: *join_all.induct*)
 case *1* **thus** *?case* **by** *simp*
next
 case (*2 t a ts*)
 thus *?case*
  apply *simp* **using** *complete_join_adj*[*of TTs t a ts n*, *simplified*] **by** *blast*
qed

lemma *complete_leaves*: *t* ∈ *trees* (*leaves as*) ⟹ *complete t* ∧ *height t* = *0*
by (*induction as*) *auto*

corollary *complete*: *complete*(*tree23_of_list as*)
by(*simp add*: *tree23_of_list_def complete_leaves complete_join_all*[*of _ 0*])

28.3 Linear running time

fun *T_join_adj* :: *'a tree23s* ⇒ *nat* **where**
T_join_adj (*TTs t1 a* (*T t2*)) = *1* |
T_join_adj (*TTs t1 a* (*TTs t2 b* (*T t3*))) = *1* |
T_join_adj (*TTs t1 a* (*TTs t2 b ts*)) = *T_join_adj ts* + *1*

fun *T_join_all* :: *'a tree23s* ⇒ *nat* **where**
T_join_all (*T t*) = *1* |
T_join_all ts = *T_join_adj ts* + *T_join_all* (*join_adj ts*) + *1*

fun *T_leaves* :: *'a list* ⇒ *nat* **where**
T_leaves [] = *1* |
T_leaves (*a* # *as*) = *T_leaves as* + *1*

definition *T_tree23_of_list* :: *'a list* ⇒ *nat* **where**

T_tree23_of_list as = *T_leaves as* + *T_join_all*(*leaves as*) + *1*

lemma *T_join_adj*: *not_T ts* ⟹ *T_join_adj ts* ≤ *len ts div 2*
by(*induction ts rule*: *T_join_adj.induct*) *auto*

lemma *len_ge_1*: *len ts* ≥ *1*
by(*cases ts*) *auto*

lemma *T_join_all*: *T_join_all ts* ≤ *2* ∗ *len ts*
proof(*induction ts rule*: *join_all.induct*)
  case *1* **thus** *?case* **by** *simp*
next
  case (*2 t a ts*)
  let *?ts* = *TTs t a ts*
  have *T_join_all ?ts* = *T_join_adj ?ts* + *T_join_all* (*join_adj ?ts*) +
1
    by *simp*
  also have ... ≤ *len ?ts div 2* + *T_join_all* (*join_adj ?ts*) + *1*
    using *T_join_adj*[*of ?ts*] **by** *simp*
  also have ... ≤ *len ?ts div 2* + *2* ∗ *len* (*join_adj ?ts*) + *1*
    using *2.IH* **by** *simp*
  also have ... ≤ *len ?ts div 2* + *2* ∗ (*len ?ts div 2*) + *1*
    using *len_join_adj_div2*[*of ?ts*] **by** *simp*
  also have ... ≤ *2* ∗ *len ?ts* **using** *len_ge_1*[*of ?ts*] **by** *linarith*
  finally show *?case* **.**
qed

lemma *T_leaves*: *T_leaves as* = *length as* + *1*
by(*induction as*) *auto*

lemma *len_leaves*: *len*(*leaves as*) = *length as* + *1*
by(*induction as*) *auto*

lemma *T_tree23_of_list*: *T_tree23_of_list as* ≤ *3*∗(*length as*) + *4*
using *T_join_all*[*of leaves as*] **by**(*simp add*: *T_tree23_of_list_def T_leaves len_leaves*)

end

29 2-3-4 Trees

theory *Tree234*
imports *Main*

begin

class *height* =
fixes *height* :: *'a* ⇒ *nat*

datatype *'a tree234* =
  Leaf (⟨⟩) |
  Node2 'a tree234 'a 'a tree234 (⟨_, _, _⟩) |
  Node3 'a tree234 'a 'a tree234 'a 'a tree234 (⟨_, _, _, _, _⟩) |
  Node4 'a tree234 'a 'a tree234 'a 'a tree234 'a 'a tree234
    (⟨_, _, _, _, _, _, _⟩)

fun *inorder* :: *'a tree234* ⇒ *'a list* **where**
inorder Leaf = [] |
inorder(*Node2 l a r*) = *inorder l* @ *a* # *inorder r* |
inorder(*Node3 l a m b r*) = *inorder l* @ *a* # *inorder m* @ *b* # *inorder r* |
inorder(*Node4 l a m b n c r*) = *inorder l* @ *a* # *inorder m* @ *b* # *inorder n* @ *c* # *inorder r*

instantiation *tree234* :: (*type*)*height*
begin

fun *height_tree234* :: *'a tree234* ⇒ *nat* **where**
height Leaf = *0* |
height (*Node2 l _ r*) = *Suc*(*max* (*height l*) (*height r*)) |
height (*Node3 l _ m _ r*) = *Suc*(*max* (*height l*) (*max* (*height m*) (*height r*))) |
height (*Node4 l _ m _ n _ r*) = *Suc*(*max* (*height l*) (*max* (*height m*) (*max* (*height n*) (*height r*))))

instance ..

end

Balanced:

fun *bal* :: *'a tree234* ⇒ *bool* **where**
bal Leaf = *True* |
bal (*Node2 l _ r*) = (*bal l* & *bal r* & *height l* = *height r*) |
bal (*Node3 l _ m _ r*) = (*bal l* & *bal m* & *bal r* & *height l* = *height m* & *height m* = *height r*) |
bal (*Node4 l _ m _ n _ r*) = (*bal l* & *bal m* & *bal n* & *bal r* & *height l* = *height m* & *height m* = *height n* & *height n* = *height r*)

end

30 2-3-4 Tree Implementation of Sets

theory *Tree234_Set*
imports
Tree234
Cmp
Set_Specs
begin

declare *sorted_wrt.simps*(*2*)[*simp del*]

30.1 Set operations on 2-3-4 trees

definition *empty* :: *'a tree234* **where**
empty = *Leaf*

fun *isin* :: *'a::linorder tree234* ⇒ *'a* ⇒ *bool* **where**
isin Leaf x = *False* |
isin (*Node2 l a r*) *x* =
(*case cmp x a of LT* ⇒ *isin l x* | *EQ* ⇒ *True* | *GT* ⇒ *isin r x*) |
isin (*Node3 l a m b r*) *x* =
(*case cmp x a of LT* ⇒ *isin l x* | *EQ* ⇒ *True* | *GT* ⇒ (*case cmp x b of*
LT ⇒ *isin m x* | *EQ* ⇒ *True* | *GT* ⇒ *isin r x*)) |
isin (*Node4 t1 a t2 b t3 c t4*) *x* =
(*case cmp x b of*
LT ⇒
(*case cmp x a of*
LT ⇒ *isin t1 x* |
EQ ⇒ *True* |
GT ⇒ *isin t2 x*) |
EQ ⇒ *True* |
GT ⇒
(*case cmp x c of*
LT ⇒ *isin t3 x* |
EQ ⇒ *True* |
GT ⇒ *isin t4 x*))

datatype *'a* up_i = T_i *'a tree234* | Up_i *'a tree234 'a 'a tree234*

fun $tree_i$ :: *'a* up_i ⇒ *'a tree234* **where**
$tree_i$ (T_i *t*) = *t* |
$tree_i$ (Up_i *l a r*) = *Node2 l a r*

```
fun ins :: 'a::linorder ⇒ 'a tree234 ⇒ 'a up_i where
ins x Leaf = Up_i Leaf x Leaf |
ins x (Node2 l a r) =
   (case cmp x a of
      LT ⇒ (case ins x l of
            T_i l' => T_i (Node2 l' a r)
          | Up_i l1 b l2 => T_i (Node3 l1 b l2 a r)) |
      EQ ⇒ T_i (Node2 l x r) |
      GT ⇒ (case ins x r of
            T_i r' => T_i (Node2 l a r')
          | Up_i r1 b r2 => T_i (Node3 l a r1 b r2))) |
ins x (Node3 l a m b r) =
   (case cmp x a of
      LT ⇒ (case ins x l of
            T_i l' => T_i (Node3 l' a m b r)
          | Up_i l1 c l2 => Up_i (Node2 l1 c l2) a (Node2 m b r)) |
      EQ ⇒ T_i (Node3 l a m b r) |
      GT ⇒ (case cmp x b of
             GT ⇒ (case ins x r of
                    T_i r' => T_i (Node3 l a m b r')
                  | Up_i r1 c r2 => Up_i (Node2 l a m) b (Node2 r1 c r2)) |
             EQ ⇒ T_i (Node3 l a m b r) |
             LT ⇒ (case ins x m of
                    T_i m' => T_i (Node3 l a m' b r)
                  | Up_i m1 c m2 => Up_i (Node2 l a m1) c (Node2 m2 b
r)))) |
ins x (Node4 t1 a t2 b t3 c t4) =
  (case cmp x b of
     LT ⇒
       (case cmp x a of
          LT ⇒
            (case ins x t1 of
               T_i t => T_i (Node4 t a t2 b t3 c t4) |
               Up_i l y r => Up_i (Node2 l y r) a (Node3 t2 b t3 c t4)) |
          EQ ⇒ T_i (Node4 t1 a t2 b t3 c t4) |
          GT ⇒
            (case ins x t2 of
               T_i t => T_i (Node4 t1 a t b t3 c t4) |
               Up_i l y r => Up_i (Node2 t1 a l) y (Node3 r b t3 c t4))) |
     EQ ⇒ T_i (Node4 t1 a t2 b t3 c t4) |
     GT ⇒
       (case cmp x c of
          LT ⇒
```

$(case\ ins\ x\ t3\ of$
$T_i\ t => T_i\ (Node4\ t1\ a\ t2\ b\ t\ c\ t4)\ |$
$Up_i\ l\ y\ r => Up_i\ (Node2\ t1\ a\ t2)\ b\ (Node3\ l\ y\ r\ c\ t4))\ |$
$EQ \Rightarrow T_i\ (Node4\ t1\ a\ t2\ b\ t3\ c\ t4)\ |$
$GT \Rightarrow$
$(case\ ins\ x\ t4\ of$
$T_i\ t => T_i\ (Node4\ t1\ a\ t2\ b\ t3\ c\ t)\ |$
$Up_i\ l\ y\ r => Up_i\ (Node2\ t1\ a\ t2)\ b\ (Node3\ t3\ c\ l\ y\ r))))$

hide_const *insert*

definition $insert :: 'a::linorder \Rightarrow 'a\ tree234 \Rightarrow 'a\ tree234$ **where**
$insert\ x\ t = tree_i(ins\ x\ t)$

datatype $'a\ up_d = T_d\ 'a\ tree234\ |\ Up_d\ 'a\ tree234$

fun $tree_d :: 'a\ up_d \Rightarrow 'a\ tree234$ **where**
$tree_d\ (T_d\ t) = t\ |$
$tree_d\ (Up_d\ t) = t$

fun $node21 :: 'a\ up_d \Rightarrow 'a \Rightarrow 'a\ tree234 \Rightarrow 'a\ up_d$ **where**
$node21\ (T_d\ l)\ a\ r = T_d(Node2\ l\ a\ r)\ |$
$node21\ (Up_d\ l)\ a\ (Node2\ lr\ b\ rr) = Up_d(Node3\ l\ a\ lr\ b\ rr)\ |$
$node21\ (Up_d\ l)\ a\ (Node3\ lr\ b\ mr\ c\ rr) = T_d(Node2\ (Node2\ l\ a\ lr)\ b\ (Node2\ mr\ c\ rr))\ |$
$node21\ (Up_d\ t1)\ a\ (Node4\ t2\ b\ t3\ c\ t4\ d\ t5) = T_d(Node2\ (Node2\ t1\ a\ t2)\ b\ (Node3\ t3\ c\ t4\ d\ t5))$

fun $node22 :: 'a\ tree234 \Rightarrow 'a \Rightarrow 'a\ up_d \Rightarrow 'a\ up_d$ **where**
$node22\ l\ a\ (T_d\ r) = T_d(Node2\ l\ a\ r)\ |$
$node22\ (Node2\ ll\ b\ rl)\ a\ (Up_d\ r) = Up_d(Node3\ ll\ b\ rl\ a\ r)\ |$
$node22\ (Node3\ ll\ b\ ml\ c\ rl)\ a\ (Up_d\ r) = T_d(Node2\ (Node2\ ll\ b\ ml)\ c\ (Node2\ rl\ a\ r))\ |$
$node22\ (Node4\ t1\ a\ t2\ b\ t3\ c\ t4)\ d\ (Up_d\ t5) = T_d(Node2\ (Node2\ t1\ a\ t2)\ b\ (Node3\ t3\ c\ t4\ d\ t5))$

fun $node31 :: 'a\ up_d \Rightarrow 'a \Rightarrow 'a\ tree234 \Rightarrow 'a \Rightarrow 'a\ tree234 \Rightarrow 'a\ up_d$ **where**
$node31\ (T_d\ t1)\ a\ t2\ b\ t3 = T_d(Node3\ t1\ a\ t2\ b\ t3)\ |$
$node31\ (Up_d\ t1)\ a\ (Node2\ t2\ b\ t3)\ c\ t4 = T_d(Node2\ (Node3\ t1\ a\ t2\ b\ t3)\ c\ t4)\ |$
$node31\ (Up_d\ t1)\ a\ (Node3\ t2\ b\ t3\ c\ t4)\ d\ t5 = T_d(Node3\ (Node2\ t1\ a\ t2)\ b\ (Node2\ t3\ c\ t4)\ d\ t5)\ |$
$node31\ (Up_d\ t1)\ a\ (Node4\ t2\ b\ t3\ c\ t4\ d\ t5)\ e\ t6 = T_d(Node3\ (Node2\ t1\ a\ t2)\ b\ (Node3\ t3\ c\ t4\ d\ t5)\ e\ t6)$

fun *node32* :: 'a tree234 ⇒ 'a ⇒ 'a up_d ⇒ 'a ⇒ 'a tree234 ⇒ 'a up_d **where**
node32 t1 a (T_d *t2*) *b t3* = T_d(*Node3 t1 a t2 b t3*) |
node32 t1 a (Up_d *t2*) *b* (*Node2 t3 c t4*) = T_d(*Node2 t1 a* (*Node3 t2 b t3 c t4*)) |
node32 t1 a (Up_d *t2*) *b* (*Node3 t3 c t4 d t5*) = T_d(*Node3 t1 a* (*Node2 t2 b t3*) *c* (*Node2 t4 d t5*)) |
node32 t1 a (Up_d *t2*) *b* (*Node4 t3 c t4 d t5 e t6*) = T_d(*Node3 t1 a* (*Node2 t2 b t3*) *c* (*Node3 t4 d t5 e t6*))

fun *node33* :: 'a tree234 ⇒ 'a ⇒ 'a tree234 ⇒ 'a ⇒ 'a up_d ⇒ 'a up_d **where**
node33 l a m b (T_d *r*) = T_d(*Node3 l a m b r*) |
node33 t1 a (*Node2 t2 b t3*) *c* (Up_d *t4*) = T_d(*Node2 t1 a* (*Node3 t2 b t3 c t4*)) |
node33 t1 a (*Node3 t2 b t3 c t4*) *d* (Up_d *t5*) = T_d(*Node3 t1 a* (*Node2 t2 b t3*) *c* (*Node2 t4 d t5*)) |
node33 t1 a (*Node4 t2 b t3 c t4 d t5*) *e* (Up_d *t6*) = T_d(*Node3 t1 a* (*Node2 t2 b t3*) *c* (*Node3 t4 d t5 e t6*))

fun *node41* :: 'a up_d ⇒ 'a ⇒ 'a tree234 ⇒ 'a ⇒ 'a tree234 ⇒ 'a ⇒ 'a tree234 ⇒ 'a up_d **where**
node41 (T_d *t1*) *a t2 b t3 c t4* = T_d(*Node4 t1 a t2 b t3 c t4*) |
node41 (Up_d *t1*) *a* (*Node2 t2 b t3*) *c t4 d t5* = T_d(*Node3* (*Node3 t1 a t2 b t3*) *c t4 d t5*) |
node41 (Up_d *t1*) *a* (*Node3 t2 b t3 c t4*) *d t5 e t6* = T_d(*Node4* (*Node2 t1 a t2*) *b* (*Node2 t3 c t4*) *d t5 e t6*) |
node41 (Up_d *t1*) *a* (*Node4 t2 b t3 c t4 d t5*) *e t6 f t7* = T_d(*Node4* (*Node2 t1 a t2*) *b* (*Node3 t3 c t4 d t5*) *e t6 f t7*)

fun *node42* :: 'a tree234 ⇒ 'a ⇒ 'a up_d ⇒ 'a ⇒ 'a tree234 ⇒ 'a ⇒ 'a tree234 ⇒ 'a up_d **where**
node42 t1 a (T_d *t2*) *b t3 c t4* = T_d(*Node4 t1 a t2 b t3 c t4*) |
node42 (*Node2 t1 a t2*) *b* (Up_d *t3*) *c t4 d t5* = T_d(*Node3* (*Node3 t1 a t2 b t3*) *c t4 d t5*) |
node42 (*Node3 t1 a t2 b t3*) *c* (Up_d *t4*) *d t5 e t6* = T_d(*Node4* (*Node2 t1 a t2*) *b* (*Node2 t3 c t4*) *d t5 e t6*) |
node42 (*Node4 t1 a t2 b t3 c t4*) *d* (Up_d *t5*) *e t6 f t7* = T_d(*Node4* (*Node2 t1 a t2*) *b* (*Node3 t3 c t4 d t5*) *e t6 f t7*)

fun *node43* :: 'a tree234 ⇒ 'a ⇒ 'a tree234 ⇒ 'a ⇒ 'a up_d ⇒ 'a ⇒ 'a tree234 ⇒ 'a up_d **where**
node43 t1 a t2 b (T_d *t3*) *c t4* = T_d(*Node4 t1 a t2 b t3 c t4*) |
node43 t1 a (*Node2 t2 b t3*) *c* (Up_d *t4*) *d t5* = T_d(*Node3 t1 a* (*Node3 t2 b t3 c t4*) *d t5*) |

node43 t1 a (*Node3 t2 b t3 c t4*) *d* (*Up*$_d$ *t5*) *e t6* = *T*$_d$(*Node4 t1 a* (*Node2 t2 b t3*) *c* (*Node2 t4 d t5*) *e t6*) |
node43 t1 a (*Node4 t2 b t3 c t4 d t5*) *e* (*Up*$_d$ *t6*) *f t7* = *T*$_d$(*Node4 t1 a* (*Node2 t2 b t3*) *c* (*Node3 t4 d t5 e t6*) *f t7*)

fun *node44* :: *'a tree234* ⇒ *'a* ⇒ *'a tree234* ⇒ *'a* ⇒ *'a tree234* ⇒ *'a* ⇒ *'a up*$_d$ ⇒ *'a up*$_d$ **where**
node44 t1 a t2 b t3 c (*T*$_d$ *t4*) = *T*$_d$(*Node4 t1 a t2 b t3 c t4*) |
node44 t1 a t2 b (*Node2 t3 c t4*) *d* (*Up*$_d$ *t5*) = *T*$_d$(*Node3 t1 a t2 b* (*Node3 t3 c t4 d t5*)) |
node44 t1 a t2 b (*Node3 t3 c t4 d t5*) *e* (*Up*$_d$ *t6*) = *T*$_d$(*Node4 t1 a t2 b* (*Node2 t3 c t4*) *d* (*Node2 t5 e t6*)) |
node44 t1 a t2 b (*Node4 t3 c t4 d t5 e t6*) *f* (*Up*$_d$ *t7*) = *T*$_d$(*Node4 t1 a t2 b* (*Node2 t3 c t4*) *d* (*Node3 t5 e t6 f t7*))

fun *split_min* :: *'a tree234* ⇒ *'a* ∗ *'a up*$_d$ **where**
split_min (*Node2 Leaf a Leaf*) = (*a*, *Up*$_d$ *Leaf*) |
split_min (*Node3 Leaf a Leaf b Leaf*) = (*a*, *T*$_d$(*Node2 Leaf b Leaf*)) |
split_min (*Node4 Leaf a Leaf b Leaf c Leaf*) = (*a*, *T*$_d$(*Node3 Leaf b Leaf c Leaf*)) |
split_min (*Node2 l a r*) = (*let* (*x*,*l'*) = *split_min l in* (*x*, *node21 l' a r*)) |
split_min (*Node3 l a m b r*) = (*let* (*x*,*l'*) = *split_min l in* (*x*, *node31 l' a m b r*)) |
split_min (*Node4 l a m b n c r*) = (*let* (*x*,*l'*) = *split_min l in* (*x*, *node41 l' a m b n c r*))

fun *del* :: *'a::linorder* ⇒ *'a tree234* ⇒ *'a up*$_d$ **where**
del k Leaf = *T*$_d$ *Leaf* |
del k (*Node2 Leaf p Leaf*) = (*if k*=*p then Up*$_d$ *Leaf else T*$_d$(*Node2 Leaf p Leaf*)) |
del k (*Node3 Leaf p Leaf q Leaf*) = *T*$_d$(*if k*=*p then Node2 Leaf q Leaf*
  else if k=*q then Node2 Leaf p Leaf else Node3 Leaf p Leaf q Leaf*) |
del k (*Node4 Leaf a Leaf b Leaf c Leaf*) =
  T$_d$(*if k*=*a then Node3 Leaf b Leaf c Leaf else*
    if k=*b then Node3 Leaf a Leaf c Leaf else*
    if k=*c then Node3 Leaf a Leaf b Leaf*
    else Node4 Leaf a Leaf b Leaf c Leaf) |
del k (*Node2 l a r*) = (*case cmp k a of*
  LT ⇒ *node21* (*del k l*) *a r* |
  GT ⇒ *node22 l a* (*del k r*) |
  EQ ⇒ *let* (*a'*,*t*) = *split_min r in node22 l a' t*) |
del k (*Node3 l a m b r*) = (*case cmp k a of*
  LT ⇒ *node31* (*del k l*) *a m b r* |
  EQ ⇒ *let* (*a'*,*m'*) = *split_min m in node32 l a' m' b r* |

$GT \Rightarrow (case\ cmp\ k\ b\ of$
$LT \Rightarrow node32\ l\ a\ (del\ k\ m)\ b\ r\ |$
$EQ \Rightarrow let\ (b',r') = split\_min\ r\ in\ node33\ l\ a\ m\ b'\ r'\ |$
$GT \Rightarrow node33\ l\ a\ m\ b\ (del\ k\ r)))\ |$
$del\ k\ (Node4\ l\ a\ m\ b\ n\ c\ r) = (case\ cmp\ k\ b\ of$
$LT \Rightarrow (case\ cmp\ k\ a\ of$
$LT \Rightarrow node41\ (del\ k\ l)\ a\ m\ b\ n\ c\ r\ |$
$EQ \Rightarrow let\ (a',m') = split\_min\ m\ in\ node42\ l\ a'\ m'\ b\ n\ c\ r\ |$
$GT \Rightarrow node42\ l\ a\ (del\ k\ m)\ b\ n\ c\ r)\ |$
$EQ \Rightarrow let\ (b',n') = split\_min\ n\ in\ node43\ l\ a\ m\ b'\ n'\ c\ r\ |$
$GT \Rightarrow (case\ cmp\ k\ c\ of$
$LT \Rightarrow node43\ l\ a\ m\ b\ (del\ k\ n)\ c\ r\ |$
$EQ \Rightarrow let\ (c',r') = split\_min\ r\ in\ node44\ l\ a\ m\ b\ n\ c'\ r'\ |$
$GT \Rightarrow node44\ l\ a\ m\ b\ n\ c\ (del\ k\ r)))$

definition $delete :: 'a::linorder \Rightarrow 'a\ tree234 \Rightarrow 'a\ tree234$ **where**
$delete\ x\ t = tree_d(del\ x\ t)$

30.2 Functional correctness

30.2.1 Functional correctness of isin:

lemma *isin_set*: $sorted(inorder\ t) \Longrightarrow isin\ t\ x = (x \in set\ (inorder\ t))$
by (*induction t*) (*auto simp*: *isin_simps*)

30.2.2 Functional correctness of insert:

lemma *inorder_ins*:
$sorted(inorder\ t) \Longrightarrow inorder(tree_i(ins\ x\ t)) = ins\_list\ x\ (inorder\ t)$
by(*induction t*) (*auto*, *auto simp*: *ins_list_simps split*!: *if_splits* $up_i.splits$)

lemma *inorder_insert*:
$sorted(inorder\ t) \Longrightarrow inorder(insert\ a\ t) = ins\_list\ a\ (inorder\ t)$
by(*simp add*: *insert_def inorder_ins*)

30.2.3 Functional correctness of delete

lemma *inorder_node21*: $height\ r > 0 \Longrightarrow$
$inorder\ (tree_d\ (node21\ l'\ a\ r)) = inorder\ (tree_d\ l')\ @\ a\ \#\ inorder\ r$
by(*induct* l' *a r rule*: *node21.induct*) *auto*

lemma *inorder_node22*: $height\ l > 0 \Longrightarrow$
$inorder\ (tree_d\ (node22\ l\ a\ r')) = inorder\ l\ @\ a\ \#\ inorder\ (tree_d\ r')$
by(*induct l a* r' *rule*: *node22.induct*) *auto*

lemma *inorder_node31*: *height m* > *0* ⟹
inorder (*tree*$_d$ (*node31 l′ a m b r*)) = *inorder* (*tree*$_d$ *l′*) @ *a* # *inorder m* @ *b* # *inorder r*
by(*induct l′ a m b r rule*: *node31.induct*) *auto*

lemma *inorder_node32*: *height r* > *0* ⟹
inorder (*tree*$_d$ (*node32 l a m′ b r*)) = *inorder l* @ *a* # *inorder* (*tree*$_d$ *m′*) @ *b* # *inorder r*
by(*induct l a m′ b r rule*: *node32.induct*) *auto*

lemma *inorder_node33*: *height m* > *0* ⟹
inorder (*tree*$_d$ (*node33 l a m b r′*)) = *inorder l* @ *a* # *inorder m* @ *b* # *inorder* (*tree*$_d$ *r′*)
by(*induct l a m b r′ rule*: *node33.induct*) *auto*

lemma *inorder_node41*: *height m* > *0* ⟹
inorder (*tree*$_d$ (*node41 l′ a m b n c r*)) = *inorder* (*tree*$_d$ *l′*) @ *a* # *inorder m* @ *b* # *inorder n* @ *c* # *inorder r*
by(*induct l′ a m b n c r rule*: *node41.induct*) *auto*

lemma *inorder_node42*: *height l* > *0* ⟹
inorder (*tree*$_d$ (*node42 l a m b n c r*)) = *inorder l* @ *a* # *inorder* (*tree*$_d$ *m*) @ *b* # *inorder n* @ *c* # *inorder r*
by(*induct l a m b n c r rule*: *node42.induct*) *auto*

lemma *inorder_node43*: *height m* > *0* ⟹
inorder (*tree*$_d$ (*node43 l a m b n c r*)) = *inorder l* @ *a* # *inorder m* @ *b* # *inorder*(*tree*$_d$ *n*) @ *c* # *inorder r*
by(*induct l a m b n c r rule*: *node43.induct*) *auto*

lemma *inorder_node44*: *height n* > *0* ⟹
inorder (*tree*$_d$ (*node44 l a m b n c r*)) = *inorder l* @ *a* # *inorder m* @ *b* # *inorder n* @ *c* # *inorder* (*tree*$_d$ *r*)
by(*induct l a m b n c r rule*: *node44.induct*) *auto*

lemmas *inorder_nodes* = *inorder_node21 inorder_node22*
inorder_node31 inorder_node32 inorder_node33
inorder_node41 inorder_node42 inorder_node43 inorder_node44

lemma *split_minD*:
split_min t = (*x*,*t′*) ⟹ *bal t* ⟹ *height t* > *0* ⟹
x # *inorder*(*tree*$_d$ *t′*) = *inorder t*
by(*induction t arbitrary*: *t′ rule*: *split_min.induct*)
(*auto simp*: *inorder_nodes split*: *prod.splits*)

lemma *inorder_del*: ⟦ *bal t* ; *sorted*(*inorder t*) ⟧ $\Longrightarrow$
  inorder(*tree*$_d$ (*del x t*)) = *del_list x* (*inorder t*)
by(*induction t rule*: *del.induct*)
 (*auto simp*: *inorder_nodes del_list_simps split_minD split*!: *if_split prod.splits*)

lemma *inorder_delete*: ⟦ *bal t* ; *sorted*(*inorder t*) ⟧ $\Longrightarrow$
  inorder(*delete x t*) = *del_list x* (*inorder t*)
by(*simp add*: *delete_def inorder_del*)

30.3 Balancedness

30.3.1 Proofs for insert

First a standard proof that *ins* preserves *bal*.

instantiation *up*$_i$ :: (*type*)*height*
begin

fun *height_up*$_i$ :: '*a up*$_i$ ⇒ *nat* **where**
height (T_i *t*) = *height t* |
height (*Up*$_i$ *l a r*) = *height l*

instance ..

end

lemma *bal_ins*: *bal t* $\Longrightarrow$ *bal* (*tree*$_i$(*ins a t*)) ∧ *height*(*ins a t*) = *height t*
by (*induct t*) (*auto split*!: *if_split up*$_i$*.split*)

Now an alternative proof (by Brian Huffman) that runs faster because two properties (balance and height) are combined in one predicate.

inductive *full* :: *nat* ⇒ '*a tree234* ⇒ *bool* **where**
full 0 Leaf |
⟦*full n l*; *full n r*⟧ $\Longrightarrow$ *full* (*Suc n*) (*Node2 l p r*) |
⟦*full n l*; *full n m*; *full n r*⟧ $\Longrightarrow$ *full* (*Suc n*) (*Node3 l p m q r*) |
⟦*full n l*; *full n m*; *full n m'*; *full n r*⟧ $\Longrightarrow$ *full* (*Suc n*) (*Node4 l p m q m' q' r*)

inductive_cases *full_elims*:
  full n Leaf
  full n (*Node2 l p r*)
  full n (*Node3 l p m q r*)
  full n (*Node4 l p m q m' q' r*)

inductive_cases *full_0_elim*: *full 0 t*
inductive_cases *full_Suc_elim*: *full* (*Suc n*) *t*

lemma *full_0_iff* [*simp*]: *full 0 t* ⟷ *t* = *Leaf*
  by (*auto elim*: *full_0_elim intro*: *full.intros*)

lemma *full_Leaf_iff* [*simp*]: *full n Leaf* ⟷ *n* = *0*
  by (*auto elim*: *full_elims intro*: *full.intros*)

lemma *full_Suc_Node2_iff* [*simp*]:
  full (*Suc n*) (*Node2 l p r*) ⟷ *full n l* ∧ *full n r*
  by (*auto elim*: *full_elims intro*: *full.intros*)

lemma *full_Suc_Node3_iff* [*simp*]:
  full (*Suc n*) (*Node3 l p m q r*) ⟷ *full n l* ∧ *full n m* ∧ *full n r*
  by (*auto elim*: *full_elims intro*: *full.intros*)

lemma *full_Suc_Node4_iff* [*simp*]:
  full (*Suc n*) (*Node4 l p m q m′ q′ r*) ⟷ *full n l* ∧ *full n m* ∧ *full n m′* ∧ *full n r*
  by (*auto elim*: *full_elims intro*: *full.intros*)

lemma *full_imp_height*: *full n t* ⟹ *height t* = *n*
  by (*induct set*: *full*, *simp_all*)

lemma *full_imp_bal*: *full n t* ⟹ *bal t*
  by (*induct set*: *full*, *auto dest*: *full_imp_height*)

lemma *bal_imp_full*: *bal t* ⟹ *full* (*height t*) *t*
  by (*induct t*, *simp_all*)

lemma *bal_iff_full*: *bal t* ⟷ (∃ *n*. *full n t*)
  by (*auto elim*!: *bal_imp_full full_imp_bal*)

The *insert* function either preserves the height of the tree, or increases it by one. The constructor returned by the *insert* function determines which: A return value of the form T_i *t* indicates that the height will be the same. A value of the form Up_i *l p r* indicates an increase in height.

primrec $full_i$ :: *nat* ⇒ ′*a* up_i ⇒ *bool* **where**
$full_i$ *n* (T_i *t*) ⟷ *full n t* |
$full_i$ *n* (Up_i *l p r*) ⟷ *full n l* ∧ *full n r*

lemma $full_i$*_ins*: *full n t* ⟹ $full_i$ *n* (*ins a t*)

by (*induct rule*: *full.induct*) (*auto*, *auto split*: up_i*.split*)

The *insert* operation preserves balance.

lemma *bal_insert*: *bal t* $\Longrightarrow$ *bal* (*insert a t*)
unfolding *bal_iff_full insert_def*
apply (*erule exE*)
apply (*drule* $full_i$*_ins* [*of* _ _ *a*])
apply (*cases ins a t*)
apply (*auto intro*: *full.intros*)
done

30.3.2 Proofs for delete

instantiation up_d :: (*type*)*height*
begin

fun *height*_up_d :: '*a* up_d $\Rightarrow$ *nat* **where**
height (T_d *t*) = *height t* |
height (Up_d *t*) = *height t* + *1*

instance ..

end

lemma *bal*_$tree_d$_*node21*:
  ⟦*bal r*; *bal* ($tree_d$ *l*); *height r* = *height l* ⟧ $\Longrightarrow$ *bal* ($tree_d$ (*node21 l a r*))
by(*induct l a r rule*: *node21.induct*) *auto*

lemma *bal*_$tree_d$_*node22*:
  ⟦*bal*($tree_d$ *r*); *bal l*; *height r* = *height l* ⟧ $\Longrightarrow$ *bal* ($tree_d$ (*node22 l a r*))
by(*induct l a r rule*: *node22.induct*) *auto*

lemma *bal*_$tree_d$_*node31*:
  ⟦ *bal* ($tree_d$ *l*); *bal m*; *bal r*; *height l* = *height r*; *height m* = *height r* ⟧
  $\Longrightarrow$ *bal* ($tree_d$ (*node31 l a m b r*))
by(*induct l a m b r rule*: *node31.induct*) *auto*

lemma *bal*_$tree_d$_*node32*:
  ⟦ *bal l*; *bal* ($tree_d$ *m*); *bal r*; *height l* = *height r*; *height m* = *height r* ⟧
  $\Longrightarrow$ *bal* ($tree_d$ (*node32 l a m b r*))
by(*induct l a m b r rule*: *node32.induct*) *auto*

lemma *bal*_$tree_d$_*node33*:
  ⟦ *bal l*; *bal m*; *bal*($tree_d$ *r*); *height l* = *height r*; *height m* = *height r* ⟧

$\Longrightarrow$ *bal* (*tree*$_d$ (*node33 l a m b r*))
by(*induct l a m b r rule*: *node33.induct*) *auto*

lemma *bal_tree*$_d$*_node41*:
⟦ *bal* (*tree*$_d$ *l*); *bal m*; *bal n*; *bal r*; *height l* = *height r*; *height m* = *height r*; *height n* = *height r* ⟧
$\Longrightarrow$ *bal* (*tree*$_d$ (*node41 l a m b n c r*))
by(*induct l a m b n c r rule*: *node41.induct*) *auto*

lemma *bal_tree*$_d$*_node42*:
⟦ *bal l*; *bal* (*tree*$_d$ *m*); *bal n*; *bal r*; *height l* = *height r*; *height m* = *height r*; *height n* = *height r* ⟧
$\Longrightarrow$ *bal* (*tree*$_d$ (*node42 l a m b n c r*))
by(*induct l a m b n c r rule*: *node42.induct*) *auto*

lemma *bal_tree*$_d$*_node43*:
⟦ *bal l*; *bal m*; *bal* (*tree*$_d$ *n*); *bal r*; *height l* = *height r*; *height m* = *height r*; *height n* = *height r* ⟧
$\Longrightarrow$ *bal* (*tree*$_d$ (*node43 l a m b n c r*))
by(*induct l a m b n c r rule*: *node43.induct*) *auto*

lemma *bal_tree*$_d$*_node44*:
⟦ *bal l*; *bal m*; *bal n*; *bal* (*tree*$_d$ *r*); *height l* = *height r*; *height m* = *height r*; *height n* = *height r* ⟧
$\Longrightarrow$ *bal* (*tree*$_d$ (*node44 l a m b n c r*))
by(*induct l a m b n c r rule*: *node44.induct*) *auto*

lemmas *bals* = *bal_tree*$_d$*_node21 bal_tree*$_d$*_node22*
bal_tree$_d$*_node31 bal_tree*$_d$*_node32 bal_tree*$_d$*_node33*
bal_tree$_d$*_node41 bal_tree*$_d$*_node42 bal_tree*$_d$*_node43 bal_tree*$_d$*_node44*

lemma *height_node21*:
height r > *0* $\Longrightarrow$ *height*(*node21 l a r*) = *max* (*height l*) (*height r*) + *1*
by(*induct l a r rule*: *node21.induct*)(*simp_all add*: *max.assoc*)

lemma *height_node22*:
height l > *0* $\Longrightarrow$ *height*(*node22 l a r*) = *max* (*height l*) (*height r*) + *1*
by(*induct l a r rule*: *node22.induct*)(*simp_all add*: *max.assoc*)

lemma *height_node31*:
height m > *0* $\Longrightarrow$ *height*(*node31 l a m b r*) =
max (*height l*) (*max* (*height m*) (*height r*)) + *1*
by(*induct l a m b r rule*: *node31.induct*)(*simp_all add*: *max_def*)

lemma *height_node32*:
  height r > *0* ⟹ *height*(*node32 l a m b r*) =
    max (*height l*) (*max* (*height m*) (*height r*)) + *1*
by(*induct l a m b r rule*: *node32.induct*)(*simp_all add*: *max_def*)

lemma *height_node33*:
  height m > *0* ⟹ *height*(*node33 l a m b r*) =
    max (*height l*) (*max* (*height m*) (*height r*)) + *1*
by(*induct l a m b r rule*: *node33.induct*)(*simp_all add*: *max_def*)

lemma *height_node41*:
  height m > *0* ⟹ *height*(*node41 l a m b n c r*) =
    max (*height l*) (*max* (*height m*) (*max* (*height n*) (*height r*))) + *1*
by(*induct l a m b n c r rule*: *node41.induct*)(*simp_all add*: *max_def*)

lemma *height_node42*:
  height l > *0* ⟹ *height*(*node42 l a m b n c r*) =
    max (*height l*) (*max* (*height m*) (*max* (*height n*) (*height r*))) + *1*
by(*induct l a m b n c r rule*: *node42.induct*)(*simp_all add*: *max_def*)

lemma *height_node43*:
  height m > *0* ⟹ *height*(*node43 l a m b n c r*) =
    max (*height l*) (*max* (*height m*) (*max* (*height n*) (*height r*))) + *1*
by(*induct l a m b n c r rule*: *node43.induct*)(*simp_all add*: *max_def*)

lemma *height_node44*:
  height n > *0* ⟹ *height*(*node44 l a m b n c r*) =
    max (*height l*) (*max* (*height m*) (*max* (*height n*) (*height r*))) + *1*
by(*induct l a m b n c r rule*: *node44.induct*)(*simp_all add*: *max_def*)

lemmas *heights* = *height_node21 height_node22*
  height_node31 height_node32 height_node33
  height_node41 height_node42 height_node43 height_node44

lemma *height_split_min*:
  split_min t = (*x*, *t'*) ⟹ *height t* > *0* ⟹ *bal t* ⟹ *height t'* = *height t*
by(*induct t arbitrary*: *x t' rule*: *split_min.induct*)
  (*auto simp*: *heights split*: *prod.splits*)

lemma *height_del*: *bal t* ⟹ *height*(*del x t*) = *height t*
by(*induction x t rule*: *del.induct*)
  (*auto simp add*: *heights height_split_min split*!: *if_split prod.split*)

lemma *bal_split_min*:

⟦ *split_min t* = (*x*, *t'*); *bal t*; *height t* > *0* ⟧ ⟹ *bal* (*tree*$_d$ *t'*)
by(*induct t arbitrary*: *x t' rule*: *split_min.induct*)
(*auto simp*: *heights height_split_min bals split*: *prod.splits*)

lemma *bal_tree*$_d$*_del*: *bal t* ⟹ *bal*(*tree*$_d$(*del x t*))
by(*induction x t rule*: *del.induct*)
(*auto simp*: *bals bal_split_min height_del height_split_min split*!: *if_split prod.split*)

corollary *bal_delete*: *bal t* ⟹ *bal*(*delete x t*)
by(*simp add*: *delete_def bal_tree*$_d$*_del*)

30.4 Overall Correctness

interpretation *S*: *Set_by_Ordered*
where *empty* = *empty* **and** *isin* = *isin* **and** *insert* = *insert* **and** *delete* = *delete*
and *inorder* = *inorder* **and** *inv* = *bal*
proof (*standard*, *goal_cases*)
case *2* **thus** *?case* **by**(*simp add*: *isin_set*)
next
case *3* **thus** *?case* **by**(*simp add*: *inorder_insert*)
next
case *4* **thus** *?case* **by**(*simp add*: *inorder_delete*)
next
case *6* **thus** *?case* **by**(*simp add*: *bal_insert*)
next
case *7* **thus** *?case* **by**(*simp add*: *bal_delete*)
qed (*simp add*: *empty_def*)+

end

31 2-3-4 Tree Implementation of Maps

theory *Tree234_Map*
imports
Tree234_Set
Map_Specs
begin

31.1 Map operations on 2-3-4 trees

fun *lookup* :: (*'a*::*linorder* ∗ *'b*) *tree234* ⇒ *'a* ⇒ *'b option* **where**
lookup Leaf x = *None* |

```
lookup (Node2 l (a,b) r) x = (case cmp x a of
  LT ⇒ lookup l x |
  GT ⇒ lookup r x |
  EQ ⇒ Some b) |
lookup (Node3 l (a1,b1) m (a2,b2) r) x = (case cmp x a1 of
  LT ⇒ lookup l x |
  EQ ⇒ Some b1 |
  GT ⇒ (case cmp x a2 of
          LT ⇒ lookup m x |
          EQ ⇒ Some b2 |
          GT ⇒ lookup r x)) |
lookup (Node4 t1 (a1,b1) t2 (a2,b2) t3 (a3,b3) t4) x = (case cmp x a2 of
  LT ⇒ (case cmp x a1 of
          LT ⇒ lookup t1 x | EQ ⇒ Some b1 | GT ⇒ lookup t2 x) |
  EQ ⇒ Some b2 |
  GT ⇒ (case cmp x a3 of
          LT ⇒ lookup t3 x | EQ ⇒ Some b3 | GT ⇒ lookup t4 x))
```

```
fun upd :: 'a::linorder ⇒ 'b ⇒ ('a*'b) tree234 ⇒ ('a*'b) up_i where
upd x y Leaf = Up_i Leaf (x,y) Leaf |
upd x y (Node2 l ab r) = (case cmp x (fst ab) of
   LT ⇒ (case upd x y l of
          T_i l' => T_i (Node2 l' ab r)
        | Up_i l1 ab' l2 => T_i (Node3 l1 ab' l2 ab r)) |
   EQ ⇒ T_i (Node2 l (x,y) r) |
   GT ⇒ (case upd x y r of
          T_i r' => T_i (Node2 l ab r')
        | Up_i r1 ab' r2 => T_i (Node3 l ab r1 ab' r2))) |
upd x y (Node3 l ab1 m ab2 r) = (case cmp x (fst ab1) of
   LT ⇒ (case upd x y l of
          T_i l' => T_i (Node3 l' ab1 m ab2 r)
        | Up_i l1 ab' l2 => Up_i (Node2 l1 ab' l2) ab1 (Node2 m ab2 r)) |
   EQ ⇒ T_i (Node3 l (x,y) m ab2 r) |
   GT ⇒ (case cmp x (fst ab2) of
          LT ⇒ (case upd x y m of
                  T_i m' => T_i (Node3 l ab1 m' ab2 r)
                | Up_i m1 ab' m2 => Up_i (Node2 l ab1 m1) ab' (Node2 m2
ab2 r)) |
          EQ ⇒ T_i (Node3 l ab1 m (x,y) r) |
          GT ⇒ (case upd x y r of
                  T_i r' => T_i (Node3 l ab1 m ab2 r')
                | Up_i r1 ab' r2 => Up_i (Node2 l ab1 m) ab2 (Node2 r1 ab'
r2)))) |
upd x y (Node4 t1 ab1 t2 ab2 t3 ab3 t4) = (case cmp x (fst ab2) of
```

LT ⇒ (case cmp x (fst ab1) of
LT ⇒ (case upd x y t1 of
T_i t1′ => T_i (Node4 t1′ ab1 t2 ab2 t3 ab3 t4)
| Up_i t11 q t12 => Up_i (Node2 t11 q t12) ab1 (Node3 t2 ab2 t3 ab3 t4)) |
EQ ⇒ T_i (Node4 t1 (x,y) t2 ab2 t3 ab3 t4) |
GT ⇒ (case upd x y t2 of
T_i t2′ => T_i (Node4 t1 ab1 t2′ ab2 t3 ab3 t4)
| Up_i t21 q t22 => Up_i (Node2 t1 ab1 t21) q (Node3 t22 ab2 t3 ab3 t4))) |
EQ ⇒ T_i (Node4 t1 ab1 t2 (x,y) t3 ab3 t4) |
GT ⇒ (case cmp x (fst ab3) of
LT ⇒ (case upd x y t3 of
T_i t3′ ⇒ T_i (Node4 t1 ab1 t2 ab2 t3′ ab3 t4)
| Up_i t31 q t32 => Up_i (Node2 t1 ab1 t2) ab24 (Node3 t31 q t32 ab3 t4)) |
EQ ⇒ T_i (Node4 t1 ab1 t2 ab2 t3 (x,y) t4) |
GT ⇒ (case upd x y t4 of
T_i t4′ => T_i (Node4 t1 ab1 t2 ab2 t3 ab3 t4′)
| Up_i t41 q t42 => Up_i (Node2 t1 ab1 t2) ab2 (Node3 t3 ab3 t41 q t42))))

definition *update :: ′a::linorder ⇒ ′b ⇒ (′a∗′b) tree234 ⇒ (′a∗′b) tree234* **where**
update x y t = $tree_i$(upd x y t)

fun *del :: ′a::linorder ⇒ (′a∗′b) tree234 ⇒ (′a∗′b) up_d* **where**
del x Leaf = T_d Leaf |
del x (Node2 Leaf ab1 Leaf) = (if x=fst ab1 then Up_d Leaf else T_d(Node2 Leaf ab1 Leaf)) |
del x (Node3 Leaf ab1 Leaf ab2 Leaf) = T_d(if x=fst ab1 then Node2 Leaf ab2 Leaf
else if x=fst ab2 then Node2 Leaf ab1 Leaf else Node3 Leaf ab1 Leaf ab2 Leaf) |
del x (Node4 Leaf ab1 Leaf ab2 Leaf ab3 Leaf) =
T_d(if x = fst ab1 then Node3 Leaf ab2 Leaf ab3 Leaf else
if x = fst ab2 then Node3 Leaf ab1 Leaf ab3 Leaf else
if x = fst ab3 then Node3 Leaf ab1 Leaf ab2 Leaf
else Node4 Leaf ab1 Leaf ab2 Leaf ab3 Leaf) |
del x (Node2 l ab1 r) = (case cmp x (fst ab1) of
LT ⇒ node21 (del x l) ab1 r |
GT ⇒ node22 l ab1 (del x r) |
EQ ⇒ let (ab1′,t) = split_min r in node22 l ab1′ t) |
del x (Node3 l ab1 m ab2 r) = (case cmp x (fst ab1) of

LT ⇒ *node31* (*del x l*) *ab1 m ab2 r* |
EQ ⇒ *let* (*ab1*′,*m*′) = *split_min m in node32 l ab1*′ *m*′ *ab2 r* |
GT ⇒ (*case cmp x* (*fst ab2*) *of*
LT ⇒ *node32 l ab1* (*del x m*) *ab2 r* |
EQ ⇒ *let* (*ab2*′,*r*′) = *split_min r in node33 l ab1 m ab2*′ *r*′ |
GT ⇒ *node33 l ab1 m ab2* (*del x r*))) |
del x (*Node4 t1 ab1 t2 ab2 t3 ab3 t4*) = (*case cmp x* (*fst ab2*) *of*
LT ⇒ (*case cmp x* (*fst ab1*) *of*
LT ⇒ *node41* (*del x t1*) *ab1 t2 ab2 t3 ab3 t4* |
EQ ⇒ *let* (*ab*′,*t2*′) = *split_min t2 in node42 t1 ab*′ *t2*′ *ab2 t3 ab3*
t4 |
GT ⇒ *node42 t1 ab1* (*del x t2*) *ab2 t3 ab3 t4*) |
EQ ⇒ *let* (*ab*′,*t3*′) = *split_min t3 in node43 t1 ab1 t2 ab*′ *t3*′ *ab3 t4* |
GT ⇒ (*case cmp x* (*fst ab3*) *of*
LT ⇒ *node43 t1 ab1 t2 ab2* (*del x t3*) *ab3 t4* |
EQ ⇒ *let* (*ab*′,*t4*′) = *split_min t4 in node44 t1 ab1 t2 ab2 t3 ab*′
t4′ |
GT ⇒ *node44 t1 ab1 t2 ab2 t3 ab3* (*del x t4*)))

definition *delete* :: ′*a*::*linorder* ⇒ (′*a*∗′*b*) *tree234* ⇒ (′*a*∗′*b*) *tree234* **where**
delete x t = $tree_d$(*del x t*)

31.2 Functional correctness

lemma *lookup_map_of*:
sorted1(*inorder t*) ⟹ *lookup t x* = *map_of* (*inorder t*) *x*
by (*induction t*) (*auto simp*: *map_of_simps split*: *option.split*)

lemma *inorder_upd*:
sorted1(*inorder t*) ⟹ *inorder*($tree_i$(*upd a b t*)) = *upd_list a b* (*inorder t*)
by(*induction t*)
(*auto simp*: *upd_list_simps*, *auto simp*: *upd_list_simps split*: up_i*.splits*)

lemma *inorder_update*:
sorted1(*inorder t*) ⟹ *inorder*(*update a b t*) = *upd_list a b* (*inorder t*)
by(*simp add*: *update_def inorder_upd*)

lemma *inorder_del*: ⟦ *bal t* ; *sorted1*(*inorder t*) ⟧ ⟹
inorder($tree_d$ (*del x t*)) = *del_list x* (*inorder t*)
by(*induction t rule*: *del.induct*)
(*auto simp*: *del_list_simps inorder_nodes split_minD split*!: *if_splits prod.splits*)

lemma *inorder_delete*: ⟦ *bal t* ; *sorted1*(*inorder t*) ⟧ ⟹
inorder(*delete x t*) = *del_list x* (*inorder t*)
by(*simp add*: *delete_def inorder_del*)

31.3 Balancedness

lemma *bal_upd*: *bal t* ⟹ *bal* ($tree_i$(*upd x y t*)) ∧ *height*(*upd x y t*) = *height t*
by (*induct t*) (*auto*, *auto split*!: *if_split* up_i.*split*)

lemma *bal_update*: *bal t* ⟹ *bal* (*update x y t*)
by (*simp add*: *update_def bal_upd*)

lemma *height_del*: *bal t* ⟹ *height*(*del x t*) = *height t*
by(*induction x t rule*: *del.induct*)
(*auto simp add*: *heights height_split_min split*!: *if_split prod.split*)

lemma *bal_tree*$_d$*_del*: *bal t* ⟹ *bal*($tree_d$(*del x t*))
by(*induction x t rule*: *del.induct*)
(*auto simp*: *bals bal_split_min height_del height_split_min split*!: *if_split prod.split*)

corollary *bal_delete*: *bal t* ⟹ *bal*(*delete x t*)
by(*simp add*: *delete_def bal_tree*$_d$*_del*)

31.4 Overall Correctness

interpretation *M*: *Map_by_Ordered*
where *empty* = *empty* **and** *lookup* = *lookup* **and** *update* = *update* **and** *delete* = *delete*
and *inorder* = *inorder* **and** *inv* = *bal*
proof (*standard*, *goal_cases*)
case *2* **thus** *?case* **by**(*simp add*: *lookup_map_of*)
next
case *3* **thus** *?case* **by**(*simp add*: *inorder_update*)
next
case *4* **thus** *?case* **by**(*simp add*: *inorder_delete*)
next
case *6* **thus** *?case* **by**(*simp add*: *bal_update*)
next
case *7* **thus** *?case* **by**(*simp add*: *bal_delete*)
qed (*simp add*: *empty_def*)+

end

32 1-2 Brother Tree Implementation of Sets

theory *Brother12_Set*
imports
  Cmp
  Set_Specs
  HOL−Number_Theory.Fib
begin

32.1 Data Type and Operations

```
datatype 'a bro =
  N0 |
  N1 'a bro |
  N2 'a bro 'a 'a bro |

  L2 'a |
  N3 'a bro 'a 'a bro 'a 'a bro

definition empty :: 'a bro where
empty = N0

fun inorder :: 'a bro ⇒ 'a list where
inorder N0 = [] |
inorder (N1 t) = inorder t |
inorder (N2 l a r) = inorder l @ a # inorder r |
inorder (L2 a) = [a] |
inorder (N3 t1 a1 t2 a2 t3) = inorder t1 @ a1 # inorder t2 @ a2 # inorder
t3

fun isin :: 'a bro ⇒ 'a::linorder ⇒ bool where
isin N0 x = False |
isin (N1 t) x = isin t x |
isin (N2 l a r) x =
  (case cmp x a of
     LT ⇒ isin l x |
     EQ ⇒ True |
     GT ⇒ isin r x)

fun n1 :: 'a bro ⇒ 'a bro where
n1 (L2 a) = N2 N0 a N0 |
n1 (N3 t1 a1 t2 a2 t3) = N2 (N2 t1 a1 t2) a2 (N1 t3) |
n1 t = N1 t
```

hide_const (**open**) *insert*

locale *insert*
begin

fun *n2* :: *'a bro* ⇒ *'a* ⇒ *'a bro* ⇒ *'a bro* **where**
n2 (*L2 a1*) *a2 t* = *N3 N0 a1 N0 a2 t* |
n2 (*N3 t1 a1 t2 a2 t3*) *a3* (*N1 t4*) = *N2* (*N2 t1 a1 t2*) *a2* (*N2 t3 a3 t4*) |
n2 (*N3 t1 a1 t2 a2 t3*) *a3 t4* = *N3* (*N2 t1 a1 t2*) *a2* (*N1 t3*) *a3 t4* |
n2 t1 a1 (*L2 a2*) = *N3 t1 a1 N0 a2 N0* |
n2 (*N1 t1*) *a1* (*N3 t2 a2 t3 a3 t4*) = *N2* (*N2 t1 a1 t2*) *a2* (*N2 t3 a3 t4*) |
n2 t1 a1 (*N3 t2 a2 t3 a3 t4*) = *N3 t1 a1* (*N1 t2*) *a2* (*N2 t3 a3 t4*) |
n2 t1 a t2 = *N2 t1 a t2*

fun *ins* :: *'a::linorder* ⇒ *'a bro* ⇒ *'a bro* **where**
ins x N0 = *L2 x* |
ins x (*N1 t*) = *n1* (*ins x t*) |
ins x (*N2 l a r*) =
  (*case cmp x a of*
    LT ⇒ *n2* (*ins x l*) *a r* |
    EQ ⇒ *N2 l a r* |
    GT ⇒ *n2 l a* (*ins x r*))

fun *tree* :: *'a bro* ⇒ *'a bro* **where**
tree (*L2 a*) = *N2 N0 a N0* |
tree (*N3 t1 a1 t2 a2 t3*) = *N2* (*N2 t1 a1 t2*) *a2* (*N1 t3*) |
tree t = *t*

definition *insert* :: *'a::linorder* ⇒ *'a bro* ⇒ *'a bro* **where**
insert x t = *tree*(*ins x t*)

end

locale *delete*
begin

fun *n2* :: *'a bro* ⇒ *'a* ⇒ *'a bro* ⇒ *'a bro* **where**
n2 (*N1 t1*) *a1* (*N1 t2*) = *N1* (*N2 t1 a1 t2*) |
n2 (*N1* (*N1 t1*)) *a1* (*N2* (*N1 t2*) *a2* (*N2 t3 a3 t4*)) =
  N1 (*N2* (*N2 t1 a1 t2*) *a2* (*N2 t3 a3 t4*)) |
n2 (*N1* (*N1 t1*)) *a1* (*N2* (*N2 t2 a2 t3*) *a3* (*N1 t4*)) =
  N1 (*N2* (*N2 t1 a1 t2*) *a2* (*N2 t3 a3 t4*)) |
n2 (*N1* (*N1 t1*)) *a1* (*N2* (*N2 t2 a2 t3*) *a3* (*N2 t4 a4 t5*)) =
  N2 (*N2* (*N1 t1*) *a1* (*N2 t2 a2 t3*)) *a3* (*N1* (*N2 t4 a4 t5*)) |

```
n2 (N2 (N1 t1) a1 (N2 t2 a2 t3)) a3 (N1 (N1 t4)) =
  N1 (N2 (N2 t1 a1 t2) a2 (N2 t3 a3 t4)) |
n2 (N2 (N2 t1 a1 t2) a2 (N1 t3)) a3 (N1 (N1 t4)) =
  N1 (N2 (N2 t1 a1 t2) a2 (N2 t3 a3 t4)) |
n2 (N2 (N2 t1 a1 t2) a2 (N2 t3 a3 t4)) a5 (N1 (N1 t5)) =
  N2 (N1 (N2 t1 a1 t2)) a2 (N2 (N2 t3 a3 t4) a5 (N1 t5)) |
n2 t1 a1 t2 = N2 t1 a1 t2

fun split_min :: 'a bro ⇒ ('a × 'a bro) option where
split_min N0 = None |
split_min (N1 t) =
  (case split_min t of
     None ⇒ None |
     Some (a, t') ⇒ Some (a, N1 t')) |
split_min (N2 t1 a t2) =
  (case split_min t1 of
     None ⇒ Some (a, N1 t2) |
     Some (b, t1') ⇒ Some (b, n2 t1' a t2))

fun del :: 'a::linorder ⇒ 'a bro ⇒ 'a bro where
del _ N0         = N0 |
del x (N1 t)     = N1 (del x t) |
del x (N2 l a r) =
  (case cmp x a of
     LT ⇒ n2 (del x l) a r |
     GT ⇒ n2 l a (del x r) |
     EQ ⇒ (case split_min r of
             None ⇒ N1 l |
             Some (b, r') ⇒ n2 l b r'))

fun tree :: 'a bro ⇒ 'a bro where
tree (N1 t) = t |
tree t = t

definition delete :: 'a::linorder ⇒ 'a bro ⇒ 'a bro where
delete a t = tree (del a t)

end
```

32.2 Invariants

```
fun B :: nat ⇒ 'a bro set
and U :: nat ⇒ 'a bro set where
B 0 = {N0} |
```

B (*Suc h*) = { *N2 t1 a t2* | *t1 a t2*.
  t1 ∈ *B h* ∪ *U h* ∧ *t2* ∈ *B h* ∨ *t1* ∈ *B h* ∧ *t2* ∈ *B h* ∪ *U h*} |
U 0 = {} |
U (*Suc h*) = *N1* ‘ *B h*

abbreviation *T h* ≡ *B h* ∪ *U h*

fun *Bp* :: *nat* ⇒ ′*a bro set* **where**
Bp 0 = *B 0* ∪ *L2* ‘ *UNIV* |
Bp (*Suc 0*) = *B* (*Suc 0*) ∪ {*N3 N0 a N0 b N0*|*a b*. *True*} |
Bp (*Suc*(*Suc h*)) = *B* (*Suc*(*Suc h*)) ∪
  {*N3 t1 a t2 b t3* | *t1 a t2 b t3*. *t1* ∈ *B* (*Suc h*) ∧ *t2* ∈ *U* (*Suc h*) ∧ *t3* ∈ *B* (*Suc h*)}

fun *Um* :: *nat* ⇒ ′*a bro set* **where**
Um 0 = {} |
Um (*Suc h*) = *N1* ‘ *T h*

32.3 Functional Correctness Proofs

32.3.1 Proofs for isin

lemma *isin_set*:
  t ∈ *T h* ⟹ *sorted*(*inorder t*) ⟹ *isin t x* = (*x* ∈ *set*(*inorder t*))
by(*induction h arbitrary*: *t*) (*fastforce simp*: *isin_simps split*: *if_splits*)+

32.3.2 Proofs for insertion

lemma *inorder_n1*: *inorder*(*n1 t*) = *inorder t*
by(*cases t rule*: *n1.cases*) (*auto simp*: *sorted_lems*)

context *insert*
begin

lemma *inorder_n2*: *inorder*(*n2 l a r*) = *inorder l* @ *a* # *inorder r*
by(*cases* (*l*,*a*,*r*) *rule*: *n2.cases*) (*auto simp*: *sorted_lems*)

lemma *inorder_tree*: *inorder*(*tree t*) = *inorder t*
by(*cases t*) *auto*

lemma *inorder_ins*: *t* ∈ *T h* ⟹
  sorted(*inorder t*) ⟹ *inorder*(*ins a t*) = *ins_list a* (*inorder t*)
by(*induction h arbitrary*: *t*) (*auto simp*: *ins_list_simps inorder_n1 inorder_n2*)

lemma *inorder_insert*: $t \in T\ h \Longrightarrow$
sorted(*inorder t*) $\Longrightarrow$ *inorder*(*insert a t*) = *ins_list a* (*inorder t*)
by(*simp add*: *insert_def inorder_ins inorder_tree*)

end

32.3.3 Proofs for deletion

context *delete*
begin

lemma *inorder_tree*: *inorder*(*tree t*) = *inorder t*
by(*cases t*) *auto*

lemma *inorder_n2*: *inorder*(*n2 l a r*) = *inorder l* @ *a* # *inorder r*
by(*cases* (*l*,*a*,*r*) *rule*: *n2.cases*) (*auto*)

lemma *inorder_split_min*:
$t \in T\ h \Longrightarrow$ (*split_min t* = *None* $\longleftrightarrow$ *inorder t* = []) $\wedge$
(*split_min t* = *Some*(*a*,*t'*) $\longrightarrow$ *inorder t* = *a* # *inorder t'*)
by(*induction h arbitrary*: *t a t'*) (*auto simp*: *inorder_n2 split*: *option.splits*)

lemma *inorder_del*:
$t \in T\ h \Longrightarrow$ *sorted*(*inorder t*) $\Longrightarrow$ *inorder*(*del x t*) = *del_list x* (*inorder t*)
apply (*induction h arbitrary*: *t*)
apply (*auto simp*: *del_list_simps inorder_n2 split*: *option.splits*)
apply (*auto simp*: *del_list_simps inorder_n2*
inorder_split_min[*OF UnI1*] *inorder_split_min*[*OF UnI2*] *split*: *option.splits*)
done

lemma *inorder_delete*:
$t \in T\ h \Longrightarrow$ *sorted*(*inorder t*) $\Longrightarrow$ *inorder*(*delete x t*) = *del_list x* (*inorder t*)
by(*simp add*: *delete_def inorder_del inorder_tree*)

end

32.4 Invariant Proofs

32.4.1 Proofs for insertion

lemma *n1_type*: $t \in Bp\ h \Longrightarrow n1\ t \in T\ (Suc\ h)$
by(*cases h rule*: *Bp.cases*) *auto*

context *insert*
begin

lemma *tree_type*: $t \in Bp\ h \Longrightarrow tree\ t \in B\ h \cup B\ (Suc\ h)$
by(*cases h rule*: *Bp.cases*) *auto*

lemma *n2_type*:
$(t1 \in Bp\ h \wedge t2 \in T\ h \longrightarrow n2\ t1\ a\ t2 \in Bp\ (Suc\ h)) \wedge$
$(t1 \in T\ h \wedge t2 \in Bp\ h \longrightarrow n2\ t1\ a\ t2 \in Bp\ (Suc\ h))$
apply(*cases h rule*: *Bp.cases*)
apply (*auto*)[*2*]
apply(*rule conjI impI* | *erule conjE exE imageE* | *simp* | *erule disjE*)+
done

lemma *Bp_if_B*: $t \in B\ h \Longrightarrow t \in Bp\ h$
by (*cases h rule*: *Bp.cases*) *simp_all*

An automatic proof:

lemma
$(t \in B\ h \longrightarrow ins\ x\ t \in Bp\ h) \wedge (t \in U\ h \longrightarrow ins\ x\ t \in T\ h)$
apply(*induction h arbitrary*: *t*)
apply (*simp*)
apply (*fastforce simp*: *Bp_if_B n2_type dest*: *n1_type*)
done

A detailed proof:

lemma *ins_type*:
shows $t \in B\ h \Longrightarrow ins\ x\ t \in Bp\ h$ **and** $t \in U\ h \Longrightarrow ins\ x\ t \in T\ h$
proof(*induction h arbitrary*: *t*)
case *0*
{ case *1* **thus** *?case* **by** *simp*
next
case *2* **thus** *?case* **by** *simp* **}**
next
case (*Suc h*)
{ case *1*
then obtain *t1 a t2* **where** [*simp*]: $t = N2\ t1\ a\ t2$ **and**
t1: $t1 \in T\ h$ **and** *t2*: $t2 \in T\ h$ **and** *t12*: $t1 \in B\ h \vee t2 \in B\ h$
by *auto*
have *?case* **if** $x < a$
proof −
have $n2\ (ins\ x\ t1)\ a\ t2 \in Bp\ (Suc\ h)$
proof *cases*

```
      assume t1 ∈ B h
      with t2 show ?thesis by (simp add: Suc.IH(1) n2_type)
    next
      assume t1 ∉ B h
      hence 1: t1 ∈ U h and 2: t2 ∈ B h using t1 t12 by auto
      show ?thesis by (metis Suc.IH(2)[OF 1] Bp_if_B[OF 2] n2_type)
    qed
    with ‹x < a› show ?case by simp
  qed
  moreover
  have ?case if a < x
  proof −
    have n2 t1 a (ins x t2) ∈ Bp (Suc h)
    proof cases
      assume t2 ∈ B h
      with t1 show ?thesis by (simp add: Suc.IH(1) n2_type)
    next
      assume t2 ∉ B h
      hence 1: t1 ∈ B h and 2: t2 ∈ U h using t2 t12 by auto
      show ?thesis by (metis Bp_if_B[OF 1] Suc.IH(2)[OF 2] n2_type)
    qed
    with ‹a < x› show ?case by simp
  qed
  moreover
  have ?case if x = a
  proof −
    from 1 have t ∈ Bp (Suc h) by(rule Bp_if_B)
    thus ?case using ‹x = a› by simp
  qed
  ultimately show ?case by auto
 next
  case 2 thus ?case using Suc(1) n1_type by fastforce }
qed

lemma insert_type:
 t ∈ B h ⟹ insert x t ∈ B h ∪ B (Suc h)
unfolding insert_def by (metis ins_type(1) tree_type)

end
```

32.4.2 Proofs for deletion

```
lemma B_simps[simp]:
 N1 t ∈ B h = False
```

```
  L2 y ∈ B h = False
  (N3 t1 a1 t2 a2 t3) ∈ B h = False
  N0 ∈ B h ⟷ h = 0
by (cases h, auto)+

context delete
begin

lemma n2_type1:
  ⟦t1 ∈ Um h; t2 ∈ B h⟧ ⟹ n2 t1 a t2 ∈ T (Suc h)
apply(cases h rule: Bp.cases)
apply auto[2]
apply(erule exE bexE conjE imageE | simp | erule disjE)+
done

lemma n2_type2:
  ⟦t1 ∈ B h ; t2 ∈ Um h ⟧ ⟹ n2 t1 a t2 ∈ T (Suc h)
apply(cases h rule: Bp.cases)
apply auto[2]
apply(erule exE bexE conjE imageE | simp | erule disjE)+
done

lemma n2_type3:
  ⟦t1 ∈ T h ; t2 ∈ T h ⟧ ⟹ n2 t1 a t2 ∈ T (Suc h)
apply(cases h rule: Bp.cases)
apply auto[2]
apply(erule exE bexE conjE imageE | simp | erule disjE)+
done

lemma split_minNoneN0: ⟦t ∈ B h; split_min t = None⟧ ⟹  t = N0
by (cases t) (auto split: option.splits)

lemma split_minNoneN1 : ⟦t ∈ U h; split_min t = None⟧ ⟹ t = N1 N0
by (cases h) (auto simp: split_minNoneN0  split: option.splits)

lemma split_min_type:
  t ∈ B h ⟹ split_min t = Some (a, t′) ⟹ t′ ∈ T h
  t ∈ U h ⟹ split_min t = Some (a, t′) ⟹ t′ ∈ Um h
proof (induction h arbitrary: t a t′)
  case (Suc h)
  { case 1
    then obtain t1 a t2 where [simp]: t = N2 t1 a t2 and
      t12: t1 ∈ T h t2 ∈ T h t1 ∈ B h ∨ t2 ∈ B h
      by auto
```

```
    show ?case
    proof (cases split_min t1)
      case None
      show ?thesis
      proof cases
        assume t1 ∈ B h
        with split_minNoneN0[OF this None] 1 show ?thesis by(auto)
      next
        assume t1 ∉ B h
        thus ?thesis using 1 None by (auto)
      qed
    next
      case [simp]: (Some bt')
      obtain b t1' where [simp]: bt' = (b,t1') by fastforce
      show ?thesis
      proof cases
        assume t1 ∈ B h
        from Suc.IH(1)[OF this] 1 have t1' ∈ T h by simp
        from n2_type3[OF this t12(2)] 1 show ?thesis by auto
      next
        assume t1 ∉ B h
        hence t1: t1 ∈ U h and t2: t2 ∈ B h using t12 by auto
        from Suc.IH(2)[OF t1] have t1' ∈ Um h by simp
        from n2_type1[OF this t2] 1 show ?thesis by auto
      qed
    qed
  }
  { case 2
    then obtain t1 where [simp]: t = N1 t1 and t1: t1 ∈ B h by auto
    show ?case
    proof (cases split_min t1)
      case None
      with split_minNoneN0[OF t1 None] 2 show ?thesis by(auto)
    next
      case [simp]: (Some bt')
      obtain b t1' where [simp]: bt' = (b,t1') by fastforce
      from Suc.IH(1)[OF t1] have t1' ∈ T h by simp
      thus ?thesis using 2 by auto
    qed
  }
qed auto

lemma del_type:
  t ∈ B h ⟹ del x t ∈ T h
```

```
  t ∈ U h ⟹ del x t ∈ Um h
proof (induction h arbitrary: x t)
  case (Suc h)
  { case 1
    then obtain l a r where [simp]: t = N2 l a r and
      lr: l ∈ T h r ∈ T h l ∈ B h ∨ r ∈ B h by auto
    have ?case if x < a
    proof cases
      assume l ∈ B h
      from n2_type3[OF Suc.IH(1)[OF this] lr(2)]
      show ?thesis using ‹x<a› by(simp)
    next
      assume l ∉ B h
      hence l ∈ U h r ∈ B h using lr by auto
      from n2_type1[OF Suc.IH(2)[OF this(1)] this(2)]
      show ?thesis using ‹x<a› by(simp)
    qed
    moreover
    have ?case if x > a
    proof cases
      assume r ∈ B h
      from n2_type3[OF lr(1) Suc.IH(1)[OF this]]
      show ?thesis using ‹x>a› by(simp)
    next
      assume r ∉ B h
      hence l ∈ B h r ∈ U h using lr by auto
      from n2_type2[OF this(1) Suc.IH(2)[OF this(2)]]
      show ?thesis using ‹x>a› by(simp)
    qed
    moreover
    have ?case if [simp]: x=a
    proof (cases split_min r)
      case None
      show ?thesis
      proof cases
        assume r ∈ B h
        with split_minNoneN0[OF this None] lr show ?thesis by(simp)
      next
        assume r ∉ B h
        hence r ∈ U h using lr by auto
       with split_minNoneN1[OF this None] lr(3) show ?thesis by (simp)
      qed
    next
      case [simp]: (Some br')
```

```
    obtain b r′ where [simp]: br′ = (b,r′) by fastforce
    show ?thesis
    proof cases
      assume r ∈ B h
      from split_min_type(1)[OF this] n2_type3[OF lr(1)]
      show ?thesis by simp
    next
      assume r ∉ B h
      hence l ∈ B h and r ∈ U h using lr by auto
      from split_min_type(2)[OF this(2)] n2_type2[OF this(1)]
      show ?thesis by simp
    qed
  qed
  ultimately show ?case by auto
 }
 { case 2 with Suc.IH(1) show ?case by auto }
qed auto

lemma tree_type: t ∈ T (h+1) ⟹ tree t ∈ B (h+1) ∪ B h
by(auto)

lemma delete_type: t ∈ B h ⟹ delete x t ∈ B h ∪ B(h−1)
unfolding delete_def
by (cases h) (simp, metis del_type(1) tree_type Suc_eq_plus1 diff_Suc_1)

end
```

32.5 Overall correctness

```
interpretation Set_by_Ordered
where empty = empty and isin = isin and insert = insert.insert
and delete = delete.delete and inorder = inorder and inv = λt. ∃h. t ∈
B h
proof (standard, goal_cases)
  case 2 thus ?case by(auto intro!: isin_set)
next
  case 3 thus ?case by(auto intro!: insert.inorder_insert)
next
  case 4 thus ?case by(auto intro!: delete.inorder_delete)
next
  case 6 thus ?case using insert.insert_type by blast
next
  case 7 thus ?case using delete.delete_type by blast
qed (auto simp: empty_def)
```

32.6 Height-Size Relation

By Daniel Stüwe

```
fun fib_tree :: nat ⇒ unit bro where
  fib_tree 0 = N0
| fib_tree (Suc 0) = N2 N0 () N0
| fib_tree (Suc(Suc h)) = N2 (fib_tree (h+1)) () (N1 (fib_tree h))

fun fib' :: nat ⇒ nat where
  fib' 0 = 0
| fib' (Suc 0) = 1
| fib' (Suc(Suc h)) = 1 + fib' (Suc h) + fib' h

fun size :: 'a bro ⇒ nat where
  size N0 = 0
| size (N1 t) = size t
| size (N2 t1 _ t2) = 1 + size t1 + size t2

lemma fib_tree_B: fib_tree h ∈ B h
by (induction h rule: fib_tree.induct) auto

declare [[names_short]]

lemma size_fib': size (fib_tree h) = fib' h
by (induction h rule: fib_tree.induct) auto

lemma fibfib: fib' h + 1 = fib (Suc(Suc h))
by (induction h rule: fib_tree.induct) auto

lemma B_N2_cases[consumes 1]:
assumes N2 t1 a t2 ∈ B (Suc n)
obtains
  (BB) t1 ∈ B n and t2 ∈ B n |
  (UB) t1 ∈ U n and t2 ∈ B n |
  (BU) t1 ∈ B n and t2 ∈ U n
using assms by auto

lemma size_bounded: t ∈ B h ⟹ size t ≥ size (fib_tree h)
unfolding size_fib' proof (induction h arbitrary: t rule: fib'.induct)
case (3 h t')
  note main = 3
  then obtain t1 a t2 where t': t' = N2 t1 a t2 by auto
  with main have N2 t1 a t2 ∈ B (Suc (Suc h)) by auto
  thus ?case proof (cases rule: B_N2_cases)
```

    case *BB*
    then obtain *x y z* **where** *t2*: *t2* = *N2 x y z* ∨ *t2* = *N2 z y x x* ∈ *B h*
by *auto*
     show *?thesis* **unfolding** *t′* **using** *main(1)[OF BB(1)] main(2)[OF t2(2)] t2(1)* **by** *auto*
  next
    case *UB*
    then obtain *t11* **where** *t1*: *t1* = *N1 t11 t11* ∈ *B h* **by** *auto*
   show *?thesis* **unfolding** *t′ t1(1)* **using** *main(2)[OF t1(2)] main(1)[OF UB(2)]* **by** *simp*
  next
    case *BU*
    then obtain *t22* **where** *t2*: *t2* = *N1 t22 t22* ∈ *B h* **by** *auto*
   show *?thesis* **unfolding** *t′ t2(1)* **using** *main(2)[OF t2(2)] main(1)[OF BU(1)]* **by** *simp*
  qed
qed *auto*

theorem *t* ∈ *B h* ⟹ *fib* (*h* + *2*) ≤ *size t* + *1*
using *size_bounded*
by (*simp add*: *size_fib′ fibfib[symmetric] del*: *fib.simps*)

end

33 1-2 Brother Tree Implementation of Maps

theory *Brother12_Map*
imports
  Brother12_Set
  Map_Specs
begin

fun *lookup* :: (′*a* × ′*b*) *bro* ⇒ ′*a*::*linorder* ⇒ ′*b option* **where**
lookup N0 x = *None* |
lookup (*N1 t*) *x* = *lookup t x* |
lookup (*N2 l* (*a*,*b*) *r*) *x* =
  (*case cmp x a of*
     LT ⇒ *lookup l x* |
     EQ ⇒ *Some b* |
     GT ⇒ *lookup r x*)

locale *update* = *insert*
begin

fun *upd* :: *'a::linorder* ⇒ *'b* ⇒ (*'a*×*'b*) *bro* ⇒ (*'a*×*'b*) *bro* **where**
upd x y N0 = *L2* (*x*,*y*) |
upd x y (*N1 t*) = *n1* (*upd x y t*) |
upd x y (*N2 l* (*a*,*b*) *r*) =
  (*case cmp x a of*
    LT ⇒ *n2* (*upd x y l*) (*a*,*b*) *r* |
    EQ ⇒ *N2 l* (*a*,*y*) *r* |
    GT ⇒ *n2 l* (*a*,*b*) (*upd x y r*))

definition *update* :: *'a::linorder* ⇒ *'b* ⇒ (*'a*×*'b*) *bro* ⇒ (*'a*×*'b*) *bro* **where**
update x y t = *tree*(*upd x y t*)

end

context *delete*
begin

fun *del* :: *'a::linorder* ⇒ (*'a*×*'b*) *bro* ⇒ (*'a*×*'b*) *bro* **where**
del _ N0 = *N0* |
del x (*N1 t*) = *N1* (*del x t*) |
del x (*N2 l* (*a*,*b*) *r*) =
  (*case cmp x a of*
    LT ⇒ *n2* (*del x l*) (*a*,*b*) *r* |
    GT ⇒ *n2 l* (*a*,*b*) (*del x r*) |
    EQ ⇒ (*case split_min r of*
        None ⇒ *N1 l* |
        Some (*ab*, *r'*) ⇒ *n2 l ab r'*))

definition *delete* :: *'a::linorder* ⇒ (*'a*×*'b*) *bro* ⇒ (*'a*×*'b*) *bro* **where**
delete a t = *tree* (*del a t*)

end

33.1 Functional Correctness Proofs

33.1.1 Proofs for lookup

lemma *lookup_map_of*: *t* ∈ *T h* ⟹
  sorted1(*inorder t*) ⟹ *lookup t x* = *map_of* (*inorder t*) *x*
by(*induction h arbitrary*: *t*) (*auto simp*: *map_of_simps split*: *option.splits*)

33.1.2 Proofs for update

context *update*

begin

lemma *inorder_upd*: $t \in T\ h \Longrightarrow$
  $sorted1(inorder\ t) \Longrightarrow inorder(upd\ x\ y\ t) = upd\_list\ x\ y\ (inorder\ t)$
by(*induction h arbitrary*: *t*) (*auto simp*: *upd_list_simps inorder_n1 inorder_n2*)

lemma *inorder_update*: $t \in T\ h \Longrightarrow$
  $sorted1(inorder\ t) \Longrightarrow inorder(update\ x\ y\ t) = upd\_list\ x\ y\ (inorder\ t)$
by(*simp add*: *update_def inorder_upd inorder_tree*)

end

33.1.3 Proofs for deletion

context *delete*
begin

lemma *inorder_del*:
  $t \in T\ h \Longrightarrow sorted1(inorder\ t) \Longrightarrow inorder(del\ x\ t) = del\_list\ x\ (inorder\ t)$
  apply (*induction h arbitrary*: *t*)
  apply (*auto simp*: *del_list_simps inorder_n2*)
  apply (*auto simp*: *del_list_simps inorder_n2*
    inorder_split_min[*OF UnI1*] *inorder_split_min*[*OF UnI2*] *split*: *option.splits*)
  done

lemma *inorder_delete*:
  $t \in T\ h \Longrightarrow sorted1(inorder\ t) \Longrightarrow inorder(delete\ x\ t) = del\_list\ x\ (inorder\ t)$
by(*simp add*: *delete_def inorder_del inorder_tree*)

end

33.2 Invariant Proofs

33.2.1 Proofs for update

context *update*
begin

lemma *upd_type*:
  $(t \in B\ h \longrightarrow upd\ x\ y\ t \in Bp\ h) \wedge (t \in U\ h \longrightarrow upd\ x\ y\ t \in T\ h)$
apply(*induction h arbitrary*: *t*)

```
 apply (simp)
apply (fastforce simp: Bp_if_B n2_type dest: n1_type)
done

lemma update_type:
  t ∈ B h ⟹ update x y t ∈ B h ∪ B (Suc h)
unfolding update_def by (metis upd_type tree_type)

end
```

33.2.2 Proofs for deletion

```
context delete
begin

lemma del_type:
  t ∈ B h ⟹ del x t ∈ T h
  t ∈ U h ⟹ del x t ∈ Um h
proof (induction h arbitrary: x t)
  case (Suc h)
  { case 1
    then obtain l a b r where [simp]: t = N2 l (a,b) r and
      lr: l ∈ T h r ∈ T h l ∈ B h ∨ r ∈ B h by auto
    have ?case if x < a
    proof cases
      assume l ∈ B h
      from n2_type3[OF Suc.IH(1)[OF this] lr(2)]
      show ?thesis using ‹x<a› by(simp)
    next
      assume l ∉ B h
      hence l ∈ U h r ∈ B h using lr by auto
      from n2_type1[OF Suc.IH(2)[OF this(1)] this(2)]
      show ?thesis using ‹x<a› by(simp)
    qed
    moreover
    have ?case if x > a
    proof cases
      assume r ∈ B h
      from n2_type3[OF lr(1) Suc.IH(1)[OF this]]
      show ?thesis using ‹x>a› by(simp)
    next
      assume r ∉ B h
      hence l ∈ B h r ∈ U h using lr by auto
      from n2_type2[OF this(1) Suc.IH(2)[OF this(2)]]
```

```
    show ?thesis using ‹x>a› by(simp)
  qed
  moreover
  have ?case if [simp]: x=a
  proof (cases split_min r)
    case None
    show ?thesis
    proof cases
      assume r ∈ B h
      with split_minNoneN0[OF this None] lr show ?thesis by(simp)
    next
      assume r ∉ B h
      hence r ∈ U h using lr by auto
     with split_minNoneN1[OF this None] lr(3) show ?thesis by (simp)
    qed
  next
    case [simp]: (Some br')
    obtain b r' where [simp]: br' = (b,r') by fastforce
    show ?thesis
    proof cases
      assume r ∈ B h
      from split_min_type(1)[OF this] n2_type3[OF lr(1)]
      show ?thesis by simp
    next
      assume r ∉ B h
      hence l ∈ B h and r ∈ U h using lr by auto
      from split_min_type(2)[OF this(2)] n2_type2[OF this(1)]
      show ?thesis by simp
    qed
  qed
  ultimately show ?case by auto
 }
 { case 2 with Suc.IH(1) show ?case by auto }
qed auto

lemma delete_type:
 t ∈ B h ⟹ delete x t ∈ B h ∪ B(h−1)
unfolding delete_def
by (cases h) (simp, metis del_type(1) tree_type Suc_eq_plus1 diff_Suc_1)

end
```

33.3 Overall correctness

interpretation *Map_by_Ordered*
where *empty* = *empty* **and** *lookup* = *lookup* **and** *update* = *update.update*
and *delete* = *delete.delete* **and** *inorder* = *inorder* **and** *inv* = $\lambda t.\ \exists h.\ t \in$
B h
proof (*standard*, *goal_cases*)
  case *2* **thus** *?case* **by**(*auto intro*!: *lookup_map_of*)
next
  case *3* **thus** *?case* **by**(*auto intro*!: *update.inorder_update*)
next
  case *4* **thus** *?case* **by**(*auto intro*!: *delete.inorder_delete*)
next
  case *6* **thus** *?case* **using** *update.update_type* **by** (*metis Un_iff*)
next
  case *7* **thus** *?case* **using** *delete.delete_type* **by** *blast*
qed (*auto simp*: *empty_def*)

end

34 AA Tree Implementation of Sets

theory *AA_Set*
imports
  Isin2
  Cmp
begin

type_synonym *'a aa_tree* = (*'a*∗*nat*) *tree*

definition *empty* :: *'a aa_tree* **where**
empty = *Leaf*

fun *lvl* :: *'a aa_tree* ⇒ *nat* **where**
lvl Leaf = *0* |
lvl (*Node* _ (_, *lv*) _) = *lv*

fun *invar* :: *'a aa_tree* ⇒ *bool* **where**
invar Leaf = *True* |
invar (*Node l* (*a*, *h*) *r*) =
 (*invar l* ∧ *invar r* ∧
  h = *lvl l* + *1* ∧ (*h* = *lvl r* + *1* ∨ (∃ *lr b rr*. *r* = *Node lr* (*b*,*h*) *rr* ∧ *h* =
lvl rr + *1*)))

fun *skew* :: *'a aa_tree* ⇒ *'a aa_tree* **where**
skew (*Node* (*Node t1* (*b*, *lvb*) *t2*) (*a*, *lva*) *t3*) =
(*if lva* = *lvb then Node t1* (*b*, *lvb*) (*Node t2* (*a*, *lva*) *t3*) *else Node* (*Node t1* (*b*, *lvb*) *t2*) (*a*, *lva*) *t3*) |
skew t = *t*

fun *split* :: *'a aa_tree* ⇒ *'a aa_tree* **where**
split (*Node t1* (*a*, *lva*) (*Node t2* (*b*, *lvb*) (*Node t3* (*c*, *lvc*) *t4*))) =
(*if lva* = *lvb* ∧ *lvb* = *lvc* — *lva* = *lvc* suffices
then Node (*Node t1* (*a*,*lva*) *t2*) (*b*,*lva*+*1*) (*Node t3* (*c*, *lva*) *t4*)
else Node t1 (*a*,*lva*) (*Node t2* (*b*,*lvb*) (*Node t3* (*c*,*lvc*) *t4*))) |
split t = *t*

hide_const (**open**) *insert*

fun *insert* :: *'a*::*linorder* ⇒ *'a aa_tree* ⇒ *'a aa_tree* **where**
insert x Leaf = *Node Leaf* (*x*, *1*) *Leaf* |
insert x (*Node t1* (*a*,*lv*) *t2*) =
(*case cmp x a of*
LT ⇒ *split* (*skew* (*Node* (*insert x t1*) (*a*,*lv*) *t2*)) |
GT ⇒ *split* (*skew* (*Node t1* (*a*,*lv*) (*insert x t2*))) |
EQ ⇒ *Node t1* (*x*, *lv*) *t2*)

fun *sngl* :: *'a aa_tree* ⇒ *bool* **where**
sngl Leaf = *False* |
sngl (*Node* _ _ *Leaf*) = *True* |
sngl (*Node* _ (_, *lva*) (*Node* _ (_, *lvb*) _)) = (*lva* > *lvb*)

definition *adjust* :: *'a aa_tree* ⇒ *'a aa_tree* **where**
adjust t =
(*case t of*
Node l (*x*,*lv*) *r* ⇒
(*if lvl l* >= *lv*−*1* ∧ *lvl r* >= *lv*−*1 then t else*
if lvl r < *lv*−*1* ∧ *sngl l then skew* (*Node l* (*x*,*lv*−*1*) *r*) *else*
if lvl r < *lv*−*1*
then case l of
Node t1 (*a*,*lva*) (*Node t2* (*b*,*lvb*) *t3*)
⇒ *Node* (*Node t1* (*a*,*lva*) *t2*) (*b*,*lvb*+*1*) (*Node t3* (*x*,*lv*−*1*) *r*)
else
if lvl r < *lv then split* (*Node l* (*x*,*lv*−*1*) *r*)
else
case r of
Node t1 (*b*,*lvb*) *t4* ⇒
(*case t1 of*

Node t2 (a,lva) t3
⇒ Node (Node l (x,lv−1) t2) (a,lva+1)
(split (Node t3 (b, if sngl t1 then lva else lva+1) t4)))))

In the paper, the last case of *adjust* is expressed with the help of an incorrect auxiliary function `nlvl`.

Function *split_max* below is called `dellrg` in the paper. The latter is incorrect for two reasons: `dellrg` is meant to delete the largest element but recurses on the left instead of the right subtree; the invariant is not restored.

fun *split_max* :: *'a aa_tree ⇒ 'a aa_tree ∗ 'a* **where**
split_max (Node l (a,lv) Leaf) = (l,a) |
split_max (Node l (a,lv) r) = (let (r',b) = split_max r in (adjust(Node l (a,lv) r'), b))

fun *delete* :: *'a::linorder ⇒ 'a aa_tree ⇒ 'a aa_tree* **where**
delete _ Leaf = Leaf |
delete x (Node l (a,lv) r) =
(case cmp x a of
LT ⇒ adjust (Node (delete x l) (a,lv) r) |
GT ⇒ adjust (Node l (a,lv) (delete x r)) |
EQ ⇒ (if l = Leaf then r
else let (l',b) = split_max l in adjust (Node l' (b,lv) r)))

fun *pre_adjust* **where**
pre_adjust (Node l (a,lv) r) = (invar l ∧ invar r ∧
((lv = lvl l + 1 ∧ (lv = lvl r + 1 ∨ lv = lvl r + 2 ∨ lv = lvl r ∧ sngl r)) ∨
(lv = lvl l + 2 ∧ (lv = lvl r + 1 ∨ lv = lvl r ∧ sngl r))))

declare *pre_adjust.simps [simp del]*

34.1 Auxiliary Proofs

lemma *split_case*: *split t = (case t of*
Node t1 (x,lvx) (Node t2 (y,lvy) (Node t3 (z,lvz) t4)) ⇒
(if lvx = lvy ∧ lvy = lvz
then Node (Node t1 (x,lvx) t2) (y,lvx+1) (Node t3 (z,lvx) t4)
else t)
| t ⇒ t)
by(*auto split*: *tree.split*)

lemma *skew_case*: *skew t = (case t of*
Node (Node t1 (y,lvy) t2) (x,lvx) t3 ⇒
(if lvx = lvy then Node t1 (y, lvx) (Node t2 (x,lvx) t3) else t)

| *t* ⇒ *t*)
by(*auto split*: *tree.split*)

lemma *lvl_0_iff*: *invar t* ⟹ *lvl t* = *0* ⟷ *t* = *Leaf*
by(*cases t*) *auto*

lemma *lvl_Suc_iff*: *lvl t* = *Suc n* ⟷ (∃ *l a r*. *t* = *Node l* (*a*,*Suc n*) *r*)
by(*cases t*) *auto*

lemma *lvl_skew*: *lvl* (*skew t*) = *lvl t*
by(*cases t rule*: *skew.cases*) *auto*

lemma *lvl_split*: *lvl* (*split t*) = *lvl t* ∨ *lvl* (*split t*) = *lvl t* + *1* ∧ *sngl* (*split t*)
by(*cases t rule*: *split.cases*) *auto*

lemma *invar_2Nodes*:*invar* (*Node l* (*x*,*lv*) (*Node rl* (*rx*, *rlv*) *rr*)) =
(*invar l* ∧ *invar* ⟨*rl*, (*rx*, *rlv*), *rr*⟩ ∧ *lv* = *Suc* (*lvl l*) ∧
(*lv* = *Suc rlv* ∨ *rlv* = *lv* ∧ *lv* = *Suc* (*lvl rr*)))
by *simp*

lemma *invar_NodeLeaf*[*simp*]:
invar (*Node l* (*x*,*lv*) *Leaf*) = (*invar l* ∧ *lv* = *Suc* (*lvl l*) ∧ *lv* = *Suc 0*)
by *simp*

lemma *sngl_if_invar*: *invar* (*Node l* (*a*, *n*) *r*) ⟹ *n* = *lvl r* ⟹ *sngl r*
by(*cases r rule*: *sngl.cases*) *clarsimp*+

34.2 Invariance

34.2.1 Proofs for insert

lemma *lvl_insert_aux*:
lvl (*insert x t*) = *lvl t* ∨ *lvl* (*insert x t*) = *lvl t* + *1* ∧ *sngl* (*insert x t*)
apply(*induction t*)
apply (*auto simp*: *lvl_skew*)
apply (*metis Suc_eq_plus1 lvl.simps*(*2*) *lvl_split lvl_skew*)+
done

lemma *lvl_insert*: **obtains**
(*Same*) *lvl* (*insert x t*) = *lvl t* |
(*Incr*) *lvl* (*insert x t*) = *lvl t* + *1 sngl* (*insert x t*)
using *lvl_insert_aux* **by** *blast*

lemma *lvl_insert_sngl*: *invar t* $\Longrightarrow$ *sngl t* $\Longrightarrow$ *lvl*(*insert x t*) = *lvl t*
proof (*induction t rule*: *insert.induct*)
  case (*2 x t1 a lv t2*)
  consider (*LT*) *x* < *a* | (*GT*) *x* > *a* | (*EQ*) *x* = *a*
    using *less_linear* **by** *blast*
  thus *?case* **proof** *cases*
    case *LT*
     thus *?thesis* **using** *2* **by** (*auto simp add*: *skew_case split_case split*: *tree.splits*)
  next
    case *GT*
    thus *?thesis* **using** *2*
    proof (*cases t1 rule*: *tree2_cases*)
      case *Node*
      thus *?thesis* **using** *2 GT*
        apply (*auto simp add*: *skew_case split_case split*: *tree.splits*)
        by (*metis less_not_refl2 lvl.simps*(*2*) *lvl_insert_aux n_not_Suc_n sngl.simps*(*3*))+
    qed (*auto simp add*: *lvl_0_iff*)
  qed *simp*
qed *simp*

lemma *skew_invar*: *invar t* $\Longrightarrow$ *skew t* = *t*
by(*cases t rule*: *skew.cases*) *auto*

lemma *split_invar*: *invar t* $\Longrightarrow$ *split t* = *t*
by(*cases t rule*: *split.cases*) *clarsimp*+

lemma *invar_NodeL*:
  $[\![$ *invar*(*Node l* (*x*, *n*) *r*); *invar l'*; *lvl l'* = *lvl l* $]\!]$ $\Longrightarrow$ *invar*(*Node l'* (*x*, *n*) *r*)
by(*auto*)

lemma *invar_NodeR*:
  $[\![$ *invar*(*Node l* (*x*, *n*) *r*); *n* = *lvl r* + *1*; *invar r'*; *lvl r'* = *lvl r* $]\!]$ $\Longrightarrow$ *invar*(*Node l* (*x*, *n*) *r'*)
by(*auto*)

lemma *invar_NodeR2*:
  $[\![$ *invar*(*Node l* (*x*, *n*) *r*); *sngl r'*; *n* = *lvl r* + *1*; *invar r'*; *lvl r'* = *n* $]\!]$ $\Longrightarrow$ *invar*(*Node l* (*x*, *n*) *r'*)
by(*cases r' rule*: *sngl.cases*) *clarsimp*+

```
lemma lvl_insert_incr_iff: (lvl(insert a t) = lvl t + 1) ⟷
  (∃ l x r. insert a t = Node l (x, lvl t + 1) r ∧ lvl l = lvl r)
apply(cases t rule: tree2_cases)
apply(auto simp add: skew_case split_case split: if_splits)
apply(auto split: tree.splits if_splits)
done

lemma invar_insert: invar t ⟹ invar(insert a t)
proof(induction t rule: tree2_induct)
  case N: (Node l x n r)
  hence il: invar l and ir: invar r by auto
  note iil = N.IH(1)[OF il]
  note iir = N.IH(2)[OF ir]
  let ?t = Node l (x, n) r
  have a < x ∨ a = x ∨ x < a by auto
  moreover
  have ?case if a < x
  proof (cases rule: lvl_insert[of a l])
    case (Same) thus ?thesis
      using ‹a<x› invar_NodeL[OF N.prems iil Same]
      by (simp add: skew_invar split_invar del: invar.simps)
  next
    case (Incr)
    then obtain t1 w t2 where ial[simp]: insert a l = Node t1 (w, n) t2
      using N.prems by (auto simp: lvl_Suc_iff)
    have l12: lvl t1 = lvl t2
      by (metis Incr(1) ial lvl_insert_incr_iff tree.inject)
    have insert a ?t = split(skew(Node (insert a l) (x,n) r))
      by(simp add: ‹a<x›)
     also have skew(Node (insert a l) (x,n) r) = Node t1 (w,n) (Node t2
(x,n) r)
      by(simp)
    also have invar(split ...)
    proof (cases r rule: tree2_cases)
      case Leaf
      hence l = Leaf using N.prems by(auto simp: lvl_0_iff)
      thus ?thesis using Leaf ial by simp
    next
      case [simp]: (Node t3 y m t4)
      show ?thesis
      proof cases
        assume m = n thus ?thesis using N(3) iil by(auto)
      next
        assume m ≠ n thus ?thesis using N(3) iil l12 by(auto)
```

```
      qed
    qed
    finally show ?thesis .
  qed
  moreover
  have ?case if x < a
  proof -
    from ‹invar ?t› have n = lvl r ∨ n = lvl r + 1 by auto
    thus ?case
    proof
      assume 0: n = lvl r
      have insert a ?t = split(skew(Node l (x, n) (insert a r)))
        using ‹a>x› by(auto)
      also have skew(Node l (x,n) (insert a r)) = Node l (x,n) (insert a r)
        using N.prems by(simp add: skew_case split: tree.split)
      also have invar(split ...)
      proof -
        from lvl_insert_sngl[OF ir sngl_if_invar[OF ‹invar ?t› 0], of a]
        obtain t1 y t2 where iar: insert a r = Node t1 (y,n) t2
          using N.prems 0 by (auto simp: lvl_Suc_iff)
        from N.prems iar 0 iir
        show ?thesis by (auto simp: split_case split: tree.splits)
      qed
      finally show ?thesis .
    next
      assume 1: n = lvl r + 1
      hence sngl ?t by(cases r) auto
      show ?thesis
      proof (cases rule: lvl_insert[of a r])
        case (Same)
       show ?thesis using ‹x<a› il ir invar_NodeR[OF N.prems 1 iir Same]
          by (auto simp add: skew_invar split_invar)
      next
        case (Incr)
       thus ?thesis using invar_NodeR2[OF ‹invar ?t› Incr(2) 1 iir] 1 ‹x
< a›
          by (auto simp add: skew_invar split_invar split: if_splits)
      qed
    qed
  qed
  moreover
  have a = x ⟹ ?case using N.prems by auto
  ultimately show ?case by blast
qed simp
```

34.2.2 Proofs for delete

lemma *invarL*: *ASSUMPTION*(*invar* ⟨*l*, (*a*, *lv*), *r*⟩) ⟹ *invar l*
by(*simp add*: *ASSUMPTION_def*)

lemma *invarR*: *ASSUMPTION*(*invar* ⟨*l*, (*a*,*lv*), *r*⟩) ⟹ *invar r*
by(*simp add*: *ASSUMPTION_def*)

lemma *sngl_NodeI*:
  sngl (*Node l* (*a*,*lv*) *r*) ⟹ *sngl* (*Node l′* (*a′*, *lv*) *r*)
by(*cases r rule*: *tree2_cases*) (*simp_all*)

declare *invarL*[*simp*] *invarR*[*simp*]

lemma *pre_cases*:
assumes *pre_adjust* (*Node l* (*x*,*lv*) *r*)
obtains
 (*tSngl*) *invar l* ∧ *invar r* ∧
   lv = *Suc* (*lvl r*) ∧ *lvl l* = *lvl r* |
 (*tDouble*) *invar l* ∧ *invar r* ∧
   lv = *lvl r* ∧ *Suc* (*lvl l*) = *lvl r* ∧ *sngl r* |
 (*rDown*) *invar l* ∧ *invar r* ∧
   lv = *Suc* (*Suc* (*lvl r*)) ∧ *lv* = *Suc* (*lvl l*) |
 (*lDown_tSngl*) *invar l* ∧ *invar r* ∧
   lv = *Suc* (*lvl r*) ∧ *lv* = *Suc* (*Suc* (*lvl l*)) |
 (*lDown_tDouble*) *invar l* ∧ *invar r* ∧
   lv = *lvl r* ∧ *lv* = *Suc* (*Suc* (*lvl l*)) ∧ *sngl r*
using *assms* **unfolding** *pre_adjust.simps*
by *auto*

declare *invar.simps*(*2*)[*simp del*] *invar_2Nodes*[*simp add*]

lemma *invar_adjust*:
  assumes *pre*: *pre_adjust* (*Node l* (*a*,*lv*) *r*)
  shows *invar*(*adjust* (*Node l* (*a*,*lv*) *r*))
using *pre* **proof** (*cases rule*: *pre_cases*)
  case (*tDouble*) **thus** *?thesis* **unfolding** *adjust_def* **by** (*cases r*) (*auto simp*: *invar.simps*(*2*))
next
  case (*rDown*)
  from *rDown* **obtain** *llv ll la lr* **where** *l*: *l* = *Node ll* (*la*, *llv*) *lr* **by** (*cases l*) *auto*
  from *rDown* **show** *?thesis* **unfolding** *adjust_def* **by** (*auto simp*: *l in-*

var.simps(*2*) *split*: *tree.splits*)
next
case (*lDown_tDouble*)
from *lDown_tDouble* **obtain** *rlv rr ra rl* **where** *r*: *r* = *Node rl* (*ra*, *rlv*) *rr* **by** (*cases r*) *auto*
from *lDown_tDouble* **and** *r* **obtain** *rrlv rrr rra rrl* **where**
rr :*rr* = *Node rrr* (*rra*, *rrlv*) *rrl* **by** (*cases rr*) *auto*
from *lDown_tDouble* **show** *?thesis* **unfolding** *adjust_def r rr*
apply (*cases rl rule*: *tree2_cases*) **apply** (*auto simp add*: *invar.simps*(*2*) *split*!: *if_split*)
using *lDown_tDouble* **by** (*auto simp*: *split_case lvl_0_iff elim*:*lvl.elims split*: *tree.split*)
qed (*auto simp*: *split_case invar.simps*(*2*) *adjust_def split*: *tree.splits*)

lemma *lvl_adjust*:
assumes *pre_adjust* (*Node l* (*a*,*lv*) *r*)
shows $lv = lvl\ (adjust(Node\ l\ (a,lv)\ r)) \vee lv = lvl\ (adjust(Node\ l\ (a,lv)\ r)) + 1$
using *assms*(*1*)
proof(*cases rule*: *pre_cases*)
case *lDown_tSngl* **thus** *?thesis*
using *lvl_split*[*of* ⟨*l*, (*a*, *lvl r*), *r*⟩] **by** (*auto simp*: *adjust_def*)
next
case *lDown_tDouble* **thus** *?thesis*
by (*auto simp*: *adjust_def invar.simps*(*2*) *split*: *tree.split*)
qed (*auto simp*: *adjust_def split*: *tree.splits*)

lemma *sngl_adjust*: **assumes** *pre_adjust* (*Node l* (*a*,*lv*) *r*)
sngl ⟨*l*, (*a*, *lv*), *r*⟩ *lv* = *lvl* (*adjust* ⟨*l*, (*a*, *lv*), *r*⟩)
shows *sngl* (*adjust* ⟨*l*, (*a*, *lv*), *r*⟩)
using *assms* **proof** (*cases rule*: *pre_cases*)
case *rDown*
thus *?thesis* **using** *assms*(*2*,*3*) **unfolding** *adjust_def*
by (*auto simp add*: *skew_case*) (*auto split*: *tree.split*)
qed (*auto simp*: *adjust_def skew_case split_case split*: *tree.split*)

definition *post_del t t'* ==
invar t' ∧
$(lvl\ t' = lvl\ t \vee lvl\ t' + 1 = lvl\ t) \wedge$
$(lvl\ t' = lvl\ t \wedge sngl\ t \longrightarrow sngl\ t')$

lemma *pre_adj_if_postR*:
invar⟨*lv*, (*l*, *a*), *r*⟩ ⟹ *post_del r r'* ⟹ *pre_adjust* ⟨*lv*, (*l*, *a*), *r'*⟩
by(*cases sngl r*)

```
(auto simp: pre_adjust.simps post_del_def invar.simps(2) elim: sngl.elims)

lemma pre_adj_if_postL:
  invar⟨l, (a, lv), r⟩ ⟹ post_del l l' ⟹ pre_adjust ⟨l', (b, lv), r⟩
by(cases sngl r)
 (auto simp: pre_adjust.simps post_del_def invar.simps(2) elim: sngl.elims)

lemma post_del_adjL:
  ⟦ invar⟨l, (a, lv), r⟩; pre_adjust ⟨l', (b, lv), r⟩ ⟧
  ⟹ post_del ⟨l, (a, lv), r⟩ (adjust ⟨l', (b, lv), r⟩)
unfolding post_del_def
by (metis invar_adjust lvl_adjust sngl_NodeI sngl_adjust lvl.simps(2))

lemma post_del_adjR:
assumes invar⟨l, (a,lv), r⟩ pre_adjust ⟨l, (a,lv), r'⟩ post_del r r'
shows post_del ⟨l, (a,lv), r⟩ (adjust ⟨l, (a,lv), r'⟩)
proof(unfold post_del_def, safe del: disjCI)
  let ?t = ⟨l, (a,lv), r⟩
  let ?t' = adjust ⟨l, (a,lv), r'⟩
  show invar ?t' by(rule invar_adjust[OF assms(2)])
  show lvl ?t' = lvl ?t ∨ lvl ?t' + 1 = lvl ?t
    using lvl_adjust[OF assms(2)] by auto
  show sngl ?t' if as: lvl ?t' = lvl ?t sngl ?t
  proof -
    have s: sngl ⟨l, (a,lv), r'⟩
    proof(cases r' rule: tree2_cases)
      case Leaf thus ?thesis by simp
    next
      case Node thus ?thesis using as(2) assms(1,3)
      by (cases r rule: tree2_cases) (auto simp: post_del_def)
    qed
    show ?thesis using as(1) sngl_adjust[OF assms(2) s] by simp
  qed
qed

declare prod.splits[split]

theorem post_split_max:
 ⟦ invar t; (t', x) = split_max t; t ≠ Leaf ⟧ ⟹ post_del t t'
proof (induction t arbitrary: t' rule: split_max.induct)
  case (2 l a lv rl bl rr)
  let ?r = ⟨rl, bl, rr⟩
  let ?t = ⟨l, (a, lv), ?r⟩
  from 2.prems(2) obtain r' where r': (r', x) = split_max ?r
```

```
    and [simp]: t' = adjust ⟨l, (a, lv), r'⟩ by auto
  from  2.IH[OF _ r'] ‹invar ?t› have post: post_del ?r r' by simp
  note preR = pre_adj_if_postR[OF ‹invar ?t› post]
  show ?case by (simp add: post_del_adjR[OF 2.prems(1) preR post])
qed (auto simp: post_del_def)

theorem post_delete: invar t ⟹ post_del t (delete x t)
proof (induction t rule: tree2_induct)
  case (Node l a lv r)

  let ?l' = delete x l and ?r' = delete x r
  let ?t = Node l (a,lv) r let ?t' = delete x ?t

  from Node.prems have inv_l: invar l and inv_r: invar r by (auto)

  note post_l' = Node.IH(1)[OF inv_l]
  note preL = pre_adj_if_postL[OF Node.prems post_l']

  note post_r' = Node.IH(2)[OF inv_r]
  note preR = pre_adj_if_postR[OF Node.prems post_r']

  show ?case
  proof (cases rule: linorder_cases[of x a])
    case less
    thus ?thesis using Node.prems by (simp add: post_del_adjL preL)
  next
    case greater
      thus ?thesis using Node.prems by (simp add: post_del_adjR preR
post_r')
  next
    case equal
    show ?thesis
    proof cases
      assume l = Leaf thus ?thesis using equal Node.prems
        by(auto simp: post_del_def invar.simps(2))
    next
      assume l ≠ Leaf thus ?thesis using equal
          by simp (metis Node.prems inv_l post_del_adjL post_split_max
pre_adj_if_postL)
    qed
  qed
qed (simp add: post_del_def)

declare invar_2Nodes[simp del]
```

34.3 Functional Correctness

34.3.1 Proofs for insert

lemma *inorder_split*: *inorder*(*split t*) = *inorder t*
by(*cases t rule*: *split.cases*) (*auto*)

lemma *inorder_skew*: *inorder*(*skew t*) = *inorder t*
by(*cases t rule*: *skew.cases*) (*auto*)

lemma *inorder_insert*:
sorted(*inorder t*) $\Longrightarrow$ *inorder*(*insert x t*) = *ins_list x* (*inorder t*)
by(*induction t*) (*auto simp*: *ins_list_simps inorder_split inorder_skew*)

34.3.2 Proofs for delete

lemma *inorder_adjust*: $t \neq$ *Leaf* $\Longrightarrow$ *pre_adjust t* $\Longrightarrow$ *inorder*(*adjust t*) = *inorder t*
by(*cases t*)
(*auto simp*: *adjust_def inorder_skew inorder_split invar.simps*(*2*) *pre_adjust.simps*
split: *tree.splits*)

lemma *split_maxD*:
⟦ *split_max t* = (t',*x*); $t \neq$ *Leaf*; *invar t* ⟧ $\Longrightarrow$ *inorder* t' @ [*x*] = *inorder t*
by(*induction t arbitrary*: t' *rule*: *split_max.induct*)
(*auto simp*: *sorted_lems inorder_adjust pre_adj_if_postR post_split_max*
split: *prod.splits*)

lemma *inorder_delete*:
invar t $\Longrightarrow$ *sorted*(*inorder t*) $\Longrightarrow$ *inorder*(*delete x t*) = *del_list x* (*inorder t*)
by(*induction t*)
(*auto simp*: *del_list_simps inorder_adjust pre_adj_if_postL pre_adj_if_postR*
post_split_max post_delete split_maxD split: *prod.splits*)

interpretation *S*: *Set_by_Ordered*
where *empty* = *empty* **and** *isin* = *isin* **and** *insert* = *insert* **and** *delete* = *delete*
and *inorder* = *inorder* **and** *inv* = *invar*
proof (*standard*, *goal_cases*)
case *1* **show** *?case* **by** (*simp add*: *empty_def*)
next
case *2* **thus** *?case* **by**(*simp add*: *isin_set_inorder*)

```
next
  case 3 thus ?case by(simp add: inorder_insert)
next
  case 4 thus ?case by(simp add: inorder_delete)
next
  case 5 thus ?case by(simp add: empty_def)
next
  case 6 thus ?case by(simp add: invar_insert)
next
  case 7 thus ?case using post_delete by(auto simp: post_del_def)
qed

end
```

35 AA Tree Implementation of Maps

```
theory AA_Map
imports
  AA_Set
  Lookup2
begin

fun update :: 'a::linorder ⇒ 'b ⇒ ('a*'b) aa_tree ⇒ ('a*'b) aa_tree where
update x y Leaf = Node Leaf ((x,y), 1) Leaf |
update x y (Node t1 ((a,b), lv) t2) =
  (case cmp x a of
     LT ⇒ split (skew (Node (update x y t1) ((a,b), lv) t2)) |
     GT ⇒ split (skew (Node t1 ((a,b), lv) (update x y t2))) |
     EQ ⇒ Node t1 ((x,y), lv) t2)

fun delete :: 'a::linorder ⇒ ('a*'b) aa_tree ⇒ ('a*'b) aa_tree where
delete _ Leaf = Leaf |
delete x (Node l ((a,b), lv) r) =
  (case cmp x a of
     LT ⇒ adjust (Node (delete x l) ((a,b), lv) r) |
     GT ⇒ adjust (Node l ((a,b), lv) (delete x r)) |
     EQ ⇒ (if l = Leaf then r
             else let (l',ab') = split_max l in adjust (Node l' (ab', lv) r)))
```

35.1 Invariance

35.1.1 Proofs for insert

```
lemma lvl_update_aux:
```

lvl (*update x y t*) = *lvl t* ∨ *lvl* (*update x y t*) = *lvl t* + *1* ∧ *sngl* (*update x y t*)
apply(*induction t*)
apply (*auto simp*: *lvl_skew*)
apply (*metis Suc_eq_plus1 lvl.simps*(*2*) *lvl_split lvl_skew*)+
done

lemma *lvl_update*: **obtains**
(*Same*) *lvl* (*update x y t*) = *lvl t* |
(*Incr*) *lvl* (*update x y t*) = *lvl t* + *1 sngl* (*update x y t*)
using *lvl_update_aux* **by** *fastforce*

declare *invar.simps*(*2*)[*simp*]

lemma *lvl_update_sngl*: *invar t* ⟹ *sngl t* ⟹ *lvl*(*update x y t*) = *lvl t*
proof (*induction t rule*: *update.induct*)
case (*2 x y t1 a b lv t2*)
consider (*LT*) *x* < *a* | (*GT*) *x* > *a* | (*EQ*) *x* = *a*
using *less_linear* **by** *blast*
thus *?case* **proof** *cases*
case *LT*
thus *?thesis* **using** *2* **by** (*auto simp add*: *skew_case split_case split*: *tree.splits*)
next
case *GT*
thus *?thesis* **using** *2* **proof** (*cases t1*)
case *Node*
thus *?thesis* **using** *2 GT*
apply (*auto simp add*: *skew_case split_case split*: *tree.splits*)
by (*metis less_not_refl2 lvl.simps*(*2*) *lvl_update_aux n_not_Suc_n sngl.simps*(*3*))+
qed (*auto simp add*: *lvl_0_iff*)
qed *simp*
qed *simp*

lemma *lvl_update_incr_iff*: (*lvl*(*update a b t*) = *lvl t* + *1*) ⟷
(∃ *l x r*. *update a b t* = *Node l* (*x*,*lvl t* + *1*) *r* ∧ *lvl l* = *lvl r*)
apply(*cases t*)
apply(*auto simp add*: *skew_case split_case split*: *if_splits*)
apply(*auto split*: *tree.splits if_splits*)
done

lemma *invar_update*: *invar t* ⟹ *invar*(*update a b t*)
proof(*induction t rule*: *tree2_induct*)

```
  case N: (Node l xy n r)
  hence il: invar l and ir: invar r by auto
  note iil = N.IH(1)[OF il]
  note iir = N.IH(2)[OF ir]
  obtain x y where [simp]: xy = (x,y) by fastforce
  let ?t = Node l (xy, n) r
  have a < x ∨ a = x ∨ x < a by auto
  moreover
  have ?case if a < x
  proof (cases rule: lvl_update[of a b l])
    case (Same) thus ?thesis
      using ‹a<x› invar_NodeL[OF N.prems iil Same]
      by (simp add: skew_invar split_invar del: invar.simps)
  next
    case (Incr)
    then obtain t1 w t2 where ial[simp]: update a b l = Node t1 (w, n) t2
      using N.prems by (auto simp: lvl_Suc_iff)
    have l12: lvl t1 = lvl t2
      by (metis Incr(1) ial lvl_update_incr_iff tree.inject)
    have update a b ?t = split(skew(Node (update a b l) (xy, n) r))
      by(simp add: ‹a<x›)
    also have skew(Node (update a b l) (xy, n) r) = Node t1 (w, n) (Node
t2 (xy, n) r)
      by(simp)
    also have invar(split ...)
    proof (cases r rule: tree2_cases)
      case Leaf
      hence l = Leaf using N.prems by(auto simp: lvl_0_iff)
      thus ?thesis using Leaf ial by simp
    next
      case [simp]: (Node t3 y m t4)
      show ?thesis
      proof cases
        assume m = n thus ?thesis using N(3) iil by(auto)
      next
        assume m ≠ n thus ?thesis using N(3) iil l12 by(auto)
      qed
    qed
    finally show ?thesis .
  qed
  moreover
  have ?case if x < a
  proof -
    from ‹invar ?t› have n = lvl r ∨ n = lvl r + 1 by auto
```

```
    thus ?case
    proof
      assume 0: n = lvl r
      have update a b ?t = split(skew(Node l (xy, n) (update a b r)))
        using ‹a>x› by(auto)
     also have skew(Node l (xy, n) (update a b r)) = Node l (xy, n) (update
a b r)
        using N.prems by(simp add: skew_case split: tree.split)
      also have invar(split ... )
      proof −
        from lvl_update_sngl[OF ir sngl_if_invar[OF ‹invar ?t› 0], of a b]
        obtain t1 p t2 where iar: update a b r = Node t1 (p, n) t2
          using N.prems 0 by (auto simp: lvl_Suc_iff)
        from N.prems iar 0 iir
        show ?thesis by (auto simp: split_case split: tree.splits)
      qed
      finally show ?thesis .
    next
      assume 1: n = lvl r + 1
      hence sngl ?t by(cases r) auto
      show ?thesis
      proof (cases rule: lvl_update[of a b r])
        case (Same)
       show ?thesis using ‹x<a› il ir invar_NodeR[OF N.prems 1 iir Same]
          by (auto simp add: skew_invar split_invar)
      next
        case (Incr)
       thus ?thesis using invar_NodeR2[OF ‹invar ?t› Incr(2) 1 iir] 1 ‹x
< a›
          by (auto simp add: skew_invar split_invar split: if_splits)
      qed
    qed
  qed
  moreover
  have a = x ⟹ ?case using N.prems by auto
  ultimately show ?case by blast
qed simp
```

35.1.2 Proofs for delete

```
declare invar.simps(2)[simp del]

theorem post_delete: invar t ⟹ post_del t (delete x t)
proof (induction t rule: tree2_induct)
```

case (*Node l ab lv r*)

obtain *a b* **where** [*simp*]: $ab = (a,b)$ **by** *fastforce*

let *?l′ = delete x l* **and** *?r′ = delete x r*
let *?t = Node l (ab, lv) r* **let** *?t′ = delete x ?t*

from *Node.prems* **have** *inv_l*: *invar l* **and** *inv_r*: *invar r* **by** (*auto*)

note *post_l′ = Node.IH(1)[OF inv_l]*
note *preL = pre_adj_if_postL[OF Node.prems post_l′]*

note *post_r′ = Node.IH(2)[OF inv_r]*
note *preR = pre_adj_if_postR[OF Node.prems post_r′]*

show *?case*
proof (*cases rule*: *linorder_cases[of x a]*)
case *less*
thus *?thesis* **using** *Node.prems* **by** (*simp add*: *post_del_adjL preL*)
next
case *greater*
thus *?thesis* **using** *Node.prems preR* **by** (*simp add*: *post_del_adjR post_r′*)
next
case *equal*
show *?thesis*
proof *cases*
assume *l = Leaf* **thus** *?thesis* **using** *equal Node.prems*
by(*auto simp*: *post_del_def invar.simps(2)*)
next
assume *l ≠ Leaf* **thus** *?thesis* **using** *equal Node.prems*
by *simp* (*metis inv_l post_del_adjL post_split_max pre_adj_if_postL*)
qed
qed
qed (*simp add*: *post_del_def*)

35.2 Functional Correctness Proofs

theorem *inorder_update*:
sorted1(inorder t) ⟹ *inorder(update x y t) = upd_list x y (inorder t)*
by (*induct t*) (*auto simp*: *upd_list_simps inorder_split inorder_skew*)

theorem *inorder_delete*:
⟦*invar t*; *sorted1(inorder t)*⟧ ⟹

inorder (*delete x t*) = *del_list x* (*inorder t*)
by(*induction t*)
(*auto simp*: *del_list_simps inorder_adjust pre_adj_if_postL pre_adj_if_postR*

post_split_max post_delete split_maxD split: *prod.splits*)

interpretation *I*: *Map_by_Ordered*
where *empty* = *empty* **and** *lookup* = *lookup* **and** *update* = *update* **and** *delete* = *delete*
and *inorder* = *inorder* **and** *inv* = *invar*
proof (*standard*, *goal_cases*)
case *1* **show** *?case* **by** (*simp add*: *empty_def*)
next
case *2* **thus** *?case* **by**(*simp add*: *lookup_map_of*)
next
case *3* **thus** *?case* **by**(*simp add*: *inorder_update*)
next
case *4* **thus** *?case* **by**(*simp add*: *inorder_delete*)
next
case *5* **thus** *?case* **by**(*simp add*: *empty_def*)
next
case *6* **thus** *?case* **by**(*simp add*: *invar_update*)
next
case *7* **thus** *?case* **using** *post_delete* **by**(*auto simp*: *post_del_def*)
qed

end

36 Join-Based Implementation of Sets

theory *Set2_Join*
imports
Isin2
begin

This theory implements the set operations *insert*, *delete*, *union*, *inter*section and *diff*erence. The implementation is based on binary search trees. All operations are reduced to a single operation *join l x r* that joins two BSTs *l* and *r* and an element *x* such that $l < x < r$.

The theory is based on theory *HOL−Data_Structures.Tree2* where nodes have an additional field. This field is ignored here but it means that this theory can be instantiated with red-black trees (see theory `Set2_Join_RBT.thy`) and other balanced trees. This approach is very concrete and fixes the type of trees. Alternatively, one could assume some abstract type $'t$ of trees with suitable decomposition and recursion operators on it.

locale *Set2_Join* =
fixes *join* :: (*'a*::*linorder*∗*'b*) *tree* ⇒ *'a* ⇒ (*'a*∗*'b*) *tree* ⇒ (*'a*∗*'b*) *tree*
fixes *inv* :: (*'a*∗*'b*) *tree* ⇒ *bool*
assumes *set_join*: *set_tree* (*join l a r*) = *set_tree l* ∪ {*a*} ∪ *set_tree r*
assumes *bst_join*: *bst* (*Node l* (*a*, *b*) *r*) ⟹ *bst* (*join l a r*)
assumes *inv_Leaf*: *inv* ⟨⟩
assumes *inv_join*: ⟦ *inv l*; *inv r* ⟧ ⟹ *inv* (*join l a r*)
assumes *inv_Node*: ⟦ *inv* (*Node l* (*a*,*b*) *r*) ⟧ ⟹ *inv l* ∧ *inv r*
begin

declare *set_join* [*simp*] *Let_def*[*simp*]

36.1 *split_min*

fun *split_min* :: (*'a*∗*'b*) *tree* ⇒ *'a* × (*'a*∗*'b*) *tree* **where**
split_min (*Node l* (*a*, _) *r*) =
(*if l* = *Leaf then* (*a*,*r*) *else let* (*m*,*l'*) = *split_min l in* (*m*, *join l' a r*))

lemma *split_min_set*:
⟦ *split_min t* = (*m*,*t'*); *t* ≠ *Leaf* ⟧ ⟹ *m* ∈ *set_tree t* ∧ *set_tree t* = {*m*} ∪ *set_tree t'*
proof(*induction t arbitrary*: *t' rule*: *tree2_induct*)
case *Node* **thus** *?case* **by**(*auto split*: *prod.splits if_splits dest*: *inv_Node*)
next
case *Leaf* **thus** *?case* **by** *simp*
qed

lemma *split_min_bst*:
⟦ *split_min t* = (*m*,*t'*); *bst t*; *t* ≠ *Leaf* ⟧ ⟹ *bst t'* ∧ (∀ *x* ∈ *set_tree t'*. *m* < *x*)
proof(*induction t arbitrary*: *t' rule*: *tree2_induct*)
case *Node* **thus** *?case* **by**(*fastforce simp*: *split_min_set bst_join split*: *prod.splits if_splits*)
next
case *Leaf* **thus** *?case* **by** *simp*
qed

lemma *split_min_inv*:
⟦ *split_min t* = (*m*,*t'*); *inv t*; *t* ≠ *Leaf* ⟧ ⟹ *inv t'*
proof(*induction t arbitrary*: *t' rule*: *tree2_induct*)
case *Node* **thus** *?case* **by**(*auto simp*: *inv_join split*: *prod.splits if_splits dest*: *inv_Node*)
next

case *Leaf* **thus** *?case* **by** *simp*
qed

36.2 *join2*

fun *join2* :: ('a*'b) *tree* ⇒ ('a*'b) *tree* ⇒ ('a*'b) *tree* **where**
join2 l ⟨⟩ = *l* |
join2 l r = (*let* (*m*,*r'*) = *split_min r in join l m r'*)

lemma *set_join2*[*simp*]: *set_tree* (*join2 l r*) = *set_tree l* ∪ *set_tree r*
by(*cases r*)(*simp_all add*: *split_min_set split*: *prod.split*)

lemma *bst_join2*: ⟦ *bst l*; *bst r*; ∀ *x* ∈ *set_tree l*. ∀ *y* ∈ *set_tree r*. *x* < *y* ⟧
⟹ *bst* (*join2 l r*)
by(*cases r*)(*simp_all add*: *bst_join split_min_set split_min_bst split*: *prod.split*)

lemma *inv_join2*: ⟦ *inv l*; *inv r* ⟧ ⟹ *inv* (*join2 l r*)
by(*cases r*)(*simp_all add*: *inv_join split_min_set split_min_inv split*: *prod.split*)

36.3 *split*

fun *split* :: ('a*'b)*tree* ⇒ 'a ⇒ ('a*'b)*tree* × *bool* × ('a*'b)*tree* **where**
split Leaf k = (*Leaf*, *False*, *Leaf*) |
split (*Node l* (*a*, _) *r*) *x* =
(*case cmp x a of*
LT ⇒ *let* (*l1*,*b*,*l2*) = *split l x in* (*l1*, *b*, *join l2 a r*) |
GT ⇒ *let* (*r1*,*b*,*r2*) = *split r x in* (*join l a r1*, *b*, *r2*) |
EQ ⇒ (*l*, *True*, *r*))

lemma *split*: *split t x* = (*l*,*b*,*r*) ⟹ *bst t* ⟹
set_tree l = {*a* ∈ *set_tree t*. *a* < *x*} ∧ *set_tree r* = {*a* ∈ *set_tree t*. *x* <
a}
∧ (*b* = (*x* ∈ *set_tree t*)) ∧ *bst l* ∧ *bst r*
proof(*induction t arbitrary*: *l b r rule*: *tree2_induct*)
case *Leaf* **thus** *?case* **by** *simp*
next
case (*Node y a b z l c r*)
consider (*LT*) *l1 xin l2* **where** (*l1*,*xin*,*l2*) = *split y x*
and *split* ⟨*y*, (*a*, *b*), *z*⟩ *x* = (*l1*, *xin*, *join l2 a z*) **and** *cmp x a* = *LT*
| (*GT*) *r1 xin r2* **where** (*r1*,*xin*,*r2*) = *split z x*
and *split* ⟨*y*, (*a*, *b*), *z*⟩ *x* = (*join y a r1*, *xin*, *r2*) **and** *cmp x a* = *GT*
| (*EQ*) *split* ⟨*y*, (*a*, *b*), *z*⟩ *x* = (*y*, *True*, *z*) **and** *cmp x a* = *EQ*
by (*force split*: *cmp_val.splits prod.splits if_splits*)

thus *?case*
proof *cases*
case (*LT l1 xin l2*)
with *Node.IH*(*1*)[*OF* ‹(*l1*,*xin*,*l2*) = *split y x*›[*symmetric*]] *Node.prems*
show *?thesis* **by** (*force intro*!: *bst_join*)
next
case (*GT r1 xin r2*)
with *Node.IH*(*2*)[*OF* ‹(*r1*,*xin*,*r2*) = *split z x*›[*symmetric*]] *Node.prems*
show *?thesis* **by** (*force intro*!: *bst_join*)
next
case *EQ*
with *Node.prems* **show** *?thesis* **by** *auto*
qed
qed

lemma *split_inv*: *split t x* = (*l*,*b*,*r*) ⟹ *inv t* ⟹ *inv l* ∧ *inv r*
proof(*induction t arbitrary*: *l b r rule*: *tree2_induct*)
case *Leaf* **thus** *?case* **by** *simp*
next
case *Node*
thus *?case* **by**(*force simp*: *inv_join split*!: *prod.splits if_splits dest*!: *inv_Node*)
qed

declare *split.simps*[*simp del*]

36.4 *insert*

definition *insert* :: *'a* ⇒ (*'a*∗*'b*) *tree* ⇒ (*'a*∗*'b*) *tree* **where**
insert x t = (*let* (*l*,_,*r*) = *split t x in join l x r*)

lemma *set_tree_insert*: *bst t* ⟹ *set_tree* (*insert x t*) = {*x*} ∪ *set_tree t*
by(*auto simp add*: *insert_def split split*: *prod.split*)

lemma *bst_insert*: *bst t* ⟹ *bst* (*insert x t*)
by(*auto simp add*: *insert_def bst_join dest*: *split split*: *prod.split*)

lemma *inv_insert*: *inv t* ⟹ *inv* (*insert x t*)
by(*force simp*: *insert_def inv_join dest*: *split_inv split*: *prod.split*)

36.5 *delete*

definition *delete* :: *'a* ⇒ (*'a*∗*'b*) *tree* ⇒ (*'a*∗*'b*) *tree* **where**
delete x t = (*let* (*l*,_,*r*) = *split t x in join2 l r*)

lemma *set_tree_delete*: *bst t* ⟹ *set_tree* (*delete x t*) = *set_tree t* − {*x*}
by(*auto simp*: *delete_def split split*: *prod.split*)

lemma *bst_delete*: *bst t* ⟹ *bst* (*delete x t*)
by(*force simp add*: *delete_def intro*: *bst_join2 dest*: *split split*: *prod.split*)

lemma *inv_delete*: *inv t* ⟹ *inv* (*delete x t*)
by(*force simp*: *delete_def inv_join2 dest*: *split_inv split*: *prod.split*)

36.6 *union*

fun *union* :: (′*a*∗′*b*)*tree* ⇒ (′*a*∗′*b*)*tree* ⇒ (′*a*∗′*b*)*tree* **where**
union t1 t2 =
  (*if t1* = *Leaf then t2 else*
   if t2 = *Leaf then t1 else*
   case t1 of Node l1 (*a*, _) *r1* ⇒
   let (*l2*,_ ,*r2*) = *split t2 a*;
       l′ = *union l1 l2*; *r*′ = *union r1 r2*
   in join l′ *a r*′)

declare *union.simps* [*simp del*]

lemma *set_tree_union*: *bst t2* ⟹ *set_tree* (*union t1 t2*) = *set_tree t1* ∪
set_tree t2
proof(*induction t1 t2 rule*: *union.induct*)
  case (*1 t1 t2*)
  then show *?case*
    by (*auto simp*: *union.simps*[*of t1 t2*] *split split*: *tree.split prod.split*)
qed

lemma *bst_union*: ⟦ *bst t1*; *bst t2* ⟧ ⟹ *bst* (*union t1 t2*)
proof(*induction t1 t2 rule*: *union.induct*)
  case (*1 t1 t2*)
  thus *?case*
    by(*fastforce simp*: *union.simps*[*of t1 t2*] *set_tree_union split intro*!:
bst_join
      split: *tree.split prod.split*)
qed

lemma *inv_union*: ⟦ *inv t1*; *inv t2* ⟧ ⟹ *inv* (*union t1 t2*)
proof(*induction t1 t2 rule*: *union.induct*)
  case (*1 t1 t2*)
  thus *?case*
    by(*auto simp*:*union.simps*[*of t1 t2*] *inv_join split_inv*

split!: *tree.split prod.split dest*: *inv_Node*)
qed

36.7 *inter*

fun *inter* :: ('*a*∗'*b*)*tree* ⇒ ('*a*∗'*b*)*tree* ⇒ ('*a*∗'*b*)*tree* **where**
inter t1 t2 =
  (*if t1* = *Leaf then Leaf else*
   if t2 = *Leaf then Leaf else*
   case t1 of Node l1 (*a*, _) *r1* ⇒
   let (*l2*,*b*,*r2*) = *split t2 a*;
       l' = *inter l1 l2*; *r'* = *inter r1 r2*
   in if b then join l' a r' else join2 l' r')

declare *inter.simps* [*simp del*]

lemma *set_tree_inter*:
  ⟦ *bst t1*; *bst t2* ⟧ ⟹ *set_tree* (*inter t1 t2*) = *set_tree t1* ∩ *set_tree t2*
proof(*induction t1 t2 rule*: *inter.induct*)
  case (*1 t1 t2*)
  show *?case*
  proof (*cases t1 rule*: *tree2_cases*)
    case *Leaf* **thus** *?thesis* **by** (*simp add*: *inter.simps*)
  next
    case [*simp*]: (*Node l1 a* _ *r1*)
    show *?thesis*
    proof (*cases t2* = *Leaf*)
      case *True* **thus** *?thesis* **by** (*simp add*: *inter.simps*)
    next
      case *False*
      let *?L1* = *set_tree l1* **let** *?R1* = *set_tree r1*
      have ∗: *a* ∉ *?L1* ∪ *?R1* **using** ‹*bst t1*› **by** (*fastforce*)
      obtain *l2 b r2* **where** *sp*: *split t2 a* = (*l2*,*b*,*r2*) **using** *prod_cases3* **by** *blast*
      let *?L2* = *set_tree l2* **let** *?R2* = *set_tree r2* **let** *?A* = *if b then* {*a*} *else* {}
      have *t2*: *set_tree t2* = *?L2* ∪ *?R2* ∪ *?A* **and**
          ∗∗: *?L2* ∩ *?R2* = {} *a* ∉ *?L2* ∪ *?R2 ?L1* ∩ *?R2* = {} *?L2* ∩ *?R1* = {}
        using *split*[*OF sp*] ‹*bst t1*› ‹*bst t2*› **by** (*force*, *force*, *force*, *force*, *force*)
      have *IHl*: *set_tree* (*inter l1 l2*) = *set_tree l1* ∩ *set_tree l2*
       using *1.IH*(*1*)[*OF* _ *False* _ _ *sp*[*symmetric*]] *1.prems*(*1*,*2*) *split*[*OF sp*] **by** *simp*

have *IHr*: *set_tree* (*inter r1 r2*) = *set_tree r1* ∩ *set_tree r2*
using *1.IH*(*2*)[*OF _ False _ _ sp*[*symmetric*]] *1.prems*(*1*,*2*) *split*[*OF sp*] **by** *simp*
have *set_tree t1* ∩ *set_tree t2* = (*?L1* ∪ *?R1* ∪ {*a*}) ∩ (*?L2* ∪ *?R2* ∪ *?A*)
by(*simp add*: *t2*)
also have ... = (*?L1* ∩ *?L2*) ∪ (*?R1* ∩ *?R2*) ∪ *?A*
using ∗ ∗∗ **by** *auto*
also have ... = *set_tree* (*inter t1 t2*)
using *IHl IHr sp inter.simps*[*of t1 t2*] *False* **by**(*simp*)
finally show *?thesis* **by** *simp*
qed
qed
qed

lemma *bst_inter*: ⟦ *bst t1*; *bst t2* ⟧ ⟹ *bst* (*inter t1 t2*)
proof(*induction t1 t2 rule*: *inter.induct*)
case (*1 t1 t2*)
thus *?case*
by(*fastforce simp*: *inter.simps*[*of t1 t2*] *set_tree_inter split*
intro!: *bst_join bst_join2 split*: *tree.split prod.split*)
qed

lemma *inv_inter*: ⟦ *inv t1*; *inv t2* ⟧ ⟹ *inv* (*inter t1 t2*)
proof(*induction t1 t2 rule*: *inter.induct*)
case (*1 t1 t2*)
thus *?case*
by(*auto simp*: *inter.simps*[*of t1 t2*] *inv_join inv_join2 split_inv*
split!: *tree.split prod.split dest*: *inv_Node*)
qed

36.8 *diff*

fun *diff* :: (′*a*∗′*b*)*tree* ⇒ (′*a*∗′*b*)*tree* ⇒ (′*a*∗′*b*)*tree* **where**
diff t1 t2 =
(*if t1* = *Leaf then Leaf else*
if t2 = *Leaf then t1 else*
case t2 of Node l2 (*a*, _) *r2* ⇒
let (*l1*,_,*r1*) = *split t1 a*;
l′ = *diff l1 l2*; *r*′ = *diff r1 r2*
in join2 l′ *r*′)

declare *diff.simps* [*simp del*]

```
lemma set_tree_diff:
  ⟦ bst t1; bst t2 ⟧ ⟹ set_tree (diff t1 t2) = set_tree t1 − set_tree t2
proof(induction t1 t2 rule: diff.induct)
  case (1 t1 t2)
  show ?case
  proof (cases t2 rule: tree2_cases)
    case Leaf thus ?thesis by (simp add: diff.simps)
  next
    case [simp]: (Node l2 a _ r2)
    show ?thesis
    proof (cases t1 = Leaf)
      case True thus ?thesis by (simp add: diff.simps)
    next
      case False
      let ?L2 = set_tree l2 let ?R2 = set_tree r2
     obtain l1 b r1 where sp: split t1 a = (l1,b,r1) using prod_cases3 by
blast
      let ?L1 = set_tree l1 let ?R1 = set_tree r1 let ?A = if b then {a}
else {}
      have t1: set_tree t1 = ?L1 ∪ ?R1 ∪ ?A and
           **: a ∉ ?L1 ∪ ?R1 ?L1 ∩ ?R2 = {} ?L2 ∩ ?R1 = {}
        using split[OF sp] ‹bst t1› ‹bst t2› by (force, force, force, force)
      have IHl: set_tree (diff l1 l2) = set_tree l1 − set_tree l2
       using 1.IH(1)[OF False _ _ _ sp[symmetric]] 1.prems(1,2) split[OF
sp] by simp
      have IHr: set_tree (diff r1 r2) = set_tree r1 − set_tree r2
       using 1.IH(2)[OF False _ _ _ sp[symmetric]] 1.prems(1,2) split[OF
sp] by simp
     have set_tree t1 − set_tree t2 = (?L1 ∪ ?R1) − (?L2 ∪ ?R2 ∪ {a})
        by(simp add: t1)
      also have ... = (?L1 − ?L2) ∪ (?R1 − ?R2)
        using ** by auto
      also have ... = set_tree (diff t1 t2)
      using IHl IHr sp diff.simps[of t1 t2] False by(simp)
      finally show ?thesis by simp
    qed
  qed
qed

lemma bst_diff: ⟦ bst t1; bst t2 ⟧ ⟹ bst (diff t1 t2)
proof(induction t1 t2 rule: diff.induct)
  case (1 t1 t2)
  thus ?case
    by(fastforce simp: diff.simps[of t1 t2] set_tree_diff split
```

intro!: *bst_join bst_join2 split*: *tree.split prod.split*)
qed

lemma *inv_diff*: ⟦ *inv t1*; *inv t2* ⟧ ⟹ *inv* (*diff t1 t2*)
proof(*induction t1 t2 rule*: *diff.induct*)
case (*1 t1 t2*)
thus *?case*
by(*auto simp*: *diff.simps*[*of t1 t2*] *inv_join inv_join2 split_inv*
split!: *tree.split prod.split dest*: *inv_Node*)
qed

Locale *Set2_Join* implements locale *Set2*:

sublocale *Set2*
where *empty* = *Leaf* **and** *insert* = *insert* **and** *delete* = *delete* **and** *isin* = *isin*
and *union* = *union* **and** *inter* = *inter* **and** *diff* = *diff*
and *set* = *set_tree* **and** *invar* = $\lambda t.$ *inv t* ∧ *bst t*
proof (*standard*, *goal_cases*)
case *1* **show** *?case* **by** (*simp*)
next
case *2* **thus** *?case* **by**(*simp add*: *isin_set_tree*)
next
case *3* **thus** *?case* **by** (*simp add*: *set_tree_insert*)
next
case *4* **thus** *?case* **by** (*simp add*: *set_tree_delete*)
next
case *5* **thus** *?case* **by** (*simp add*: *inv_Leaf*)
next
case *6* **thus** *?case* **by** (*simp add*: *bst_insert inv_insert*)
next
case *7* **thus** *?case* **by** (*simp add*: *bst_delete inv_delete*)
next
case *8* **thus** *?case* **by**(*simp add*: *set_tree_union*)
next
case *9* **thus** *?case* **by**(*simp add*: *set_tree_inter*)
next
case *10* **thus** *?case* **by**(*simp add*: *set_tree_diff*)
next
case *11* **thus** *?case* **by** (*simp add*: *bst_union inv_union*)
next
case *12* **thus** *?case* **by** (*simp add*: *bst_inter inv_inter*)
next
case *13* **thus** *?case* **by** (*simp add*: *bst_diff inv_diff*)
qed

end

interpretation *unbal*: *Set2_Join*
where *join* = $\lambda l\ x\ r.$ *Node l* (*x*, ()) *r* **and** *inv* = $\lambda t.$ *True*
proof (*standard*, *goal_cases*)
  case *1* **show** *?case* **by** *simp*
next
  case *2* **thus** *?case* **by** *simp*
next
  case *3* **thus** *?case* **by** *simp*
next
  case *4* **thus** *?case* **by** *simp*
next
  case *5* **thus** *?case* **by** *simp*
qed

end

37 Join-Based Implementation of Sets via RBTs

theory *Set2_Join_RBT*
imports
  Set2_Join
  RBT_Set
begin

37.1 Code

Function *joinL* joins two trees (and an element). Precondition: *bheight l* $\leq$ *bheight r*. Method: Descend along the left spine of *r* until you find a subtree with the same *bheight* as *l*, then combine them into a new red node.

fun *joinL* :: *'a rbt* ⇒ *'a* ⇒ *'a rbt* ⇒ *'a rbt* **where**
joinL l x r =
  (*if bheight l* $\geq$ *bheight r then R l x r*
   else case r of
     B l' x' r' ⇒ *baliL* (*joinL l x l'*) *x' r'* |
     R l' x' r' ⇒ *R* (*joinL l x l'*) *x' r'*)

fun *joinR* :: *'a rbt* ⇒ *'a* ⇒ *'a rbt* ⇒ *'a rbt* **where**
joinR l x r =
  (*if bheight l* $\leq$ *bheight r then R l x r*
   else case l of

B l' x' r' ⇒ *baliR l' x'* (*joinR r' x r*) |
R l' x' r' ⇒ *R l' x'* (*joinR r' x r*))

definition *join* :: *'a rbt* ⇒ *'a* ⇒ *'a rbt* ⇒ *'a rbt* **where**
join l x r =
(*if bheight l* > *bheight r*
then paint Black (*joinR l x r*)
else if bheight l < *bheight r*
then paint Black (*joinL l x r*)
else B l x r)

declare *joinL.simps*[*simp del*]
declare *joinR.simps*[*simp del*]

37.2 Properties

37.2.1 Color and height invariants

lemma *invc2_joinL*:
⟦ *invc l*; *invc r*; *bheight l* ≤ *bheight r* ⟧ ⟹
invc2 (*joinL l x r*)
∧ (*bheight l* ≠ *bheight r* ∧ *color r* = *Black* ⟶ *invc*(*joinL l x r*))
proof (*induct l x r rule*: *joinL.induct*)
case (*1 l x r*) **thus** *?case*
by(*auto simp*: *invc_baliL invc2I joinL.simps*[*of l x r*] *split*!: *tree.splits if_splits*)
qed

lemma *invc2_joinR*:
⟦ *invc l*; *invh l*; *invc r*; *invh r*; *bheight l* ≥ *bheight r* ⟧ ⟹
invc2 (*joinR l x r*)
∧ (*bheight l* ≠ *bheight r* ∧ *color l* = *Black* ⟶ *invc*(*joinR l x r*))
proof (*induct l x r rule*: *joinR.induct*)
case (*1 l x r*) **thus** *?case*
by(*fastforce simp*: *invc_baliR invc2I joinR.simps*[*of l x r*] *split*!: *tree.splits if_splits*)
qed

lemma *bheight_joinL*:
⟦ *invh l*; *invh r*; *bheight l* ≤ *bheight r* ⟧ ⟹ *bheight* (*joinL l x r*) = *bheight r*
proof (*induct l x r rule*: *joinL.induct*)
case (*1 l x r*) **thus** *?case*
by(*auto simp*: *bheight_baliL joinL.simps*[*of l x r*] *split*!: *tree.split*)

qed

lemma *invh_joinL*:
⟦ *invh l*; *invh r*; *bheight l* ≤ *bheight r* ⟧ ⟹ *invh* (*joinL l x r*)
proof (*induct l x r rule*: *joinL.induct*)
case (*1 l x r*) **thus** *?case*
by(*auto simp*: *invh_baliL bheight_joinL joinL.simps*[*of l x r*] *split*!: *tree.split color.split*)
qed

lemma *bheight_joinR*:
⟦ *invh l*; *invh r*; *bheight l* ≥ *bheight r* ⟧ ⟹ *bheight* (*joinR l x r*) = *bheight l*
proof (*induct l x r rule*: *joinR.induct*)
case (*1 l x r*) **thus** *?case*
by(*fastforce simp*: *bheight_baliR joinR.simps*[*of l x r*] *split*!: *tree.split*)
qed

lemma *invh_joinR*:
⟦ *invh l*; *invh r*; *bheight l* ≥ *bheight r* ⟧ ⟹ *invh* (*joinR l x r*)
proof (*induct l x r rule*: *joinR.induct*)
case (*1 l x r*) **thus** *?case*
by(*fastforce simp*: *invh_baliR bheight_joinR joinR.simps*[*of l x r*]
split!: *tree.split color.split*)
qed

All invariants in one:

lemma *inv_joinL*: ⟦ *invc l*; *invc r*; *invh l*; *invh r*; *bheight l* ≤ *bheight r* ⟧
⟹ *invc2* (*joinL l x r*) ∧ (*bheight l* ≠ *bheight r* ∧ *color r* = *Black* ⟶ *invc* (*joinL l x r*))
∧ *invh* (*joinL l x r*) ∧ *bheight* (*joinL l x r*) = *bheight r*
proof (*induct l x r rule*: *joinL.induct*)
case (*1 l x r*) **thus** *?case*
by(*auto simp*: *inv_baliL invc2I joinL.simps*[*of l x r*] *split*!: *tree.splits if_splits*)
qed

lemma *inv_joinR*: ⟦ *invc l*; *invc r*; *invh l*; *invh r*; *bheight l* ≥ *bheight r* ⟧
⟹ *invc2* (*joinR l x r*) ∧ (*bheight l* ≠ *bheight r* ∧ *color l* = *Black* ⟶ *invc* (*joinR l x r*))
∧ *invh* (*joinR l x r*) ∧ *bheight* (*joinR l x r*) = *bheight l*
proof (*induct l x r rule*: *joinR.induct*)
case (*1 l x r*) **thus** *?case*
by(*auto simp*: *inv_baliR invc2I joinR.simps*[*of l x r*] *split*!: *tree.splits*

if_splits)
qed

lemma *rbt_join*: ⟦ *invc l*; *invh l*; *invc r*; *invh r* ⟧ ⟹ *rbt*(*join l x r*)
by(*simp add*: *inv_joinL inv_joinR invh_paint rbt_def color_paint_Black join_def*)

To make sure the the black height is not increased unnecessarily:

lemma *bheight_paint_Black*: *bheight*(*paint Black t*) ≤ *bheight t* + *1*
by(*cases t*) *auto*

lemma ⟦ *rbt l*; *rbt r* ⟧ ⟹ *bheight*(*join l x r*) ≤ *max* (*bheight l*) (*bheight r*) + *1*
using *bheight_paint_Black*[*of joinL l x r*] *bheight_paint_Black*[*of joinR l x r*]
bheight_joinL[*of l r x*] *bheight_joinR*[*of l r x*]
by(*auto simp*: *max_def rbt_def join_def*)

37.2.2 Inorder properties

Currently unused. Instead *Tree2.set_tree* and *Tree2.bst* properties are proved directly.

lemma *inorder_joinL*: *bheight l* ≤ *bheight r* ⟹ *inorder*(*joinL l x r*) = *inorder l* @ *x* # *inorder r*
proof(*induction l x r rule*: *joinL.induct*)
case (*1 l x r*)
thus *?case* **by**(*auto simp*: *inorder_baliL joinL.simps*[*of l x r*] *split*!: *tree.splits color.splits*)
qed

lemma *inorder_joinR*:
inorder(*joinR l x r*) = *inorder l* @ *x* # *inorder r*
proof(*induction l x r rule*: *joinR.induct*)
case (*1 l x r*)
thus *?case* **by** (*force simp*: *inorder_baliR joinR.simps*[*of l x r*] *split*!: *tree.splits color.splits*)
qed

lemma *inorder*(*join l x r*) = *inorder l* @ *x* # *inorder r*
by(*auto simp*: *inorder_joinL inorder_joinR inorder_paint join_def*
split!: *tree.splits color.splits if_splits*
dest!: *arg_cong*[**where** *f* = *inorder*])

37.2.3 Set and bst properties

lemma *set_baliL*:
set_tree(*baliL l a r*) = *set_tree l* ∪ {*a*} ∪ *set_tree r*
by(*cases* (*l*,*a*,*r*) *rule*: *baliL.cases*) (*auto*)

lemma *set_joinL*:
bheight l ≤ *bheight r* ⟹ *set_tree* (*joinL l x r*) = *set_tree l* ∪ {*x*} ∪ *set_tree r*
proof(*induction l x r rule*: *joinL.induct*)
case (*1 l x r*)
thus *?case* **by**(*auto simp*: *set_baliL joinL.simps*[*of l x r*] *split*!: *tree.splits color.splits*)
qed

lemma *set_baliR*:
set_tree(*baliR l a r*) = *set_tree l* ∪ {*a*} ∪ *set_tree r*
by(*cases* (*l*,*a*,*r*) *rule*: *baliR.cases*) (*auto*)

lemma *set_joinR*:
set_tree (*joinR l x r*) = *set_tree l* ∪ {*x*} ∪ *set_tree r*
proof(*induction l x r rule*: *joinR.induct*)
case (*1 l x r*)
thus *?case* **by**(*force simp*: *set_baliR joinR.simps*[*of l x r*] *split*!: *tree.splits color.splits*)
qed

lemma *set_paint*: *set_tree* (*paint c t*) = *set_tree t*
by (*cases t*) *auto*

lemma *set_join*: *set_tree* (*join l x r*) = *set_tree l* ∪ {*x*} ∪ *set_tree r*
by(*simp add*: *set_joinL set_joinR set_paint join_def*)

lemma *bst_baliL*:
⟦*bst l*; *bst r*; ∀ *x*∈*set_tree l*. *x* < *a*; ∀ *x*∈*set_tree r*. *a* < *x*⟧
⟹ *bst* (*baliL l a r*)
by(*cases* (*l*,*a*,*r*) *rule*: *baliL.cases*) (*auto simp*: *ball_Un*)

lemma *bst_baliR*:
⟦*bst l*; *bst r*; ∀ *x*∈*set_tree l*. *x* < *a*; ∀ *x*∈*set_tree r*. *a* < *x*⟧
⟹ *bst* (*baliR l a r*)
by(*cases* (*l*,*a*,*r*) *rule*: *baliR.cases*) (*auto simp*: *ball_Un*)

lemma *bst_joinL*:

⟦bst (Node l (a, n) r); bheight l ≤ bheight r⟧
⟹ bst (joinL l a r)
proof(induction l a r rule: joinL.induct)
case (1 l a r)
thus ?case
by(auto simp: set_baliL joinL.simps[of l a r] set_joinL ball_Un intro!: bst_baliL
split!: tree.splits color.splits)
qed

lemma bst_joinR:
⟦bst l; bst r; ∀x∈set_tree l. x < a; ∀y∈set_tree r. a < y ⟧
⟹ bst (joinR l a r)
proof(induction l a r rule: joinR.induct)
case (1 l a r)
thus ?case
by(auto simp: set_baliR joinR.simps[of l a r] set_joinR ball_Un intro!: bst_baliR
split!: tree.splits color.splits)
qed

lemma bst_paint: bst (paint c t) = bst t
by(cases t) auto

lemma bst_join:
bst (Node l (a, n) r) ⟹ bst (join l a r)
by(auto simp: bst_paint bst_joinL bst_joinR join_def)

lemma inv_join: ⟦ invc l; invh l; invc r; invh r ⟧ ⟹ invc(join l x r) ∧ invh(join l x r)
by (simp add: inv_joinL inv_joinR invh_paint join_def)

37.2.4 Interpretation of *Set2_Join* with Red-Black Tree

global_interpretation RBT: Set2_Join
where join = join **and** inv = λt. invc t ∧ invh t
defines insert_rbt = RBT.insert **and** delete_rbt = RBT.delete **and** split_rbt = RBT.split
and join2_rbt = RBT.join2 **and** split_min_rbt = RBT.split_min
proof (standard, goal_cases)
case 1 **show** ?case **by** (rule set_join)
next
case 2 **thus** ?case **by** (simp add: bst_join)
next

case *3* **show** *?case* **by** *simp*
next
case *4* **thus** *?case* **by** (*simp add*: *inv_join*)
next
case *5* **thus** *?case* **by** *simp*
qed

The invariant does not guarantee that the root node is black. This is not required to guarantee that the height is logarithmic in the size — Exercise.

end
theory *Array_Specs*
imports *Main*
begin

Array Specifications

locale *Array* =
fixes *lookup* :: *'ar* ⇒ *nat* ⇒ *'a*
fixes *update* :: *nat* ⇒ *'a* ⇒ *'ar* ⇒ *'ar*
fixes *len* :: *'ar* ⇒ *nat*
fixes *array* :: *'a list* ⇒ *'ar*

fixes *list* :: *'ar* ⇒ *'a list*
fixes *invar* :: *'ar* ⇒ *bool*

assumes *lookup*: *invar ar* ⟹ *n* < *len ar* ⟹ *lookup ar n* = *list ar* ! *n*
assumes *update*: *invar ar* ⟹ *n* < *len ar* ⟹ *list*(*update n x ar*) = (*list ar*)[*n*:=*x*]
assumes *len_array*: *invar ar* ⟹ *len ar* = *length* (*list ar*)
assumes *array*: *list* (*array xs*) = *xs*

assumes *invar_update*: *invar ar* ⟹ *n* < *len ar* ⟹ *invar*(*update n x ar*)
assumes *invar_array*: *invar*(*array xs*)

locale *Array_Flex* = *Array* +
fixes *add_lo* :: *'a* ⇒ *'ar* ⇒ *'ar*
fixes *del_lo* :: *'ar* ⇒ *'ar*
fixes *add_hi* :: *'a* ⇒ *'ar* ⇒ *'ar*
fixes *del_hi* :: *'ar* ⇒ *'ar*

assumes *add_lo*: *invar ar* ⟹ *list*(*add_lo a ar*) = *a* # *list ar*
assumes *del_lo*: *invar ar* ⟹ *list*(*del_lo ar*) = *tl* (*list ar*)
assumes *add_hi*: *invar ar* ⟹ *list*(*add_hi a ar*) = *list ar* @ [*a*]
assumes *del_hi*: *invar ar* ⟹ *list*(*del_hi ar*) = *butlast* (*list ar*)

assumes *invar_add_lo*: *invar ar* ⟹ *invar* (*add_lo a ar*)
assumes *invar_del_lo*: *invar ar* ⟹ *invar* (*del_lo ar*)
assumes *invar_add_hi*: *invar ar* ⟹ *invar* (*add_hi a ar*)
assumes *invar_del_hi*: *invar ar* ⟹ *invar* (*del_hi ar*)

end

38 Braun Trees

theory *Braun_Tree*
imports *HOL−Library.Tree_Real*
begin

Braun Trees were studied by Braun and Rem [5] and later Hoogerwoord [10].

fun *braun* :: *'a tree* ⇒ *bool* **where**
braun Leaf = *True* |
braun (*Node l x r*) = ((*size l* = *size r* ∨ *size l* = *size r* + *1*) ∧ *braun l* ∧ *braun r*)

lemma *braun_Node'*:
braun (*Node l x r*) = (*size r* ≤ *size l* ∧ *size l* ≤ *size r* + *1* ∧ *braun l* ∧ *braun r*)
by *auto*

The shape of a Braun-tree is uniquely determined by its size:

lemma *braun_unique*: ⟦ *braun* (*t1*::*unit tree*); *braun t2*; *size t1* = *size t2* ⟧ ⟹ *t1* = *t2*
proof (*induction t1 arbitrary*: *t2*)
 case *Leaf* **thus** *?case* **by** *simp*
next
 case (*Node l1 _ r1*)
 from *Node.prems*(*3*) **have** *t2* ≠ *Leaf* **by** *auto*
 then obtain *l2 x2 r2* **where** [*simp*]: *t2* = *Node l2 x2 r2* **by** (*meson neq_Leaf_iff*)
 with *Node.prems* **have** *size l1* = *size l2* ∧ *size r1* = *size r2* **by** *auto*
 thus *?case* **using** *Node.prems*(*1*,*2*) *Node.IH* **by** *auto*
qed

Braun trees are almost complete:

lemma *acomplete_if_braun*: *braun t* ⟹ *acomplete t*
proof(*induction t*)
 case *Leaf* **show** *?case* **by** (*simp add*: *acomplete_def*)
next

```
  case (Node l x r) thus ?case using acomplete_Node_if_wbal2 by force
qed
```

38.1 Numbering Nodes

We show that a tree is a Braun tree iff a parity-based numbering (*braun_indices*) of nodes yields an interval of numbers.

```
fun braun_indices :: 'a tree ⇒ nat set where
braun_indices Leaf = {} |
braun_indices (Node l _ r) = {1} ∪ (∗) 2 ‘ braun_indices l ∪ Suc ‘ (∗) 2
‘ braun_indices r

lemma braun_indices1: 0 ∉ braun_indices t
by (induction t) auto

lemma finite_braun_indices: finite(braun_indices t)
by (induction t) auto
```

One direction:

```
lemma braun_indices_if_braun: braun t ⟹ braun_indices t = {1..size
t}
proof(induction t)
  case Leaf thus ?case by simp
next
  have ∗: (∗) 2 ‘ {a..b} ∪ Suc ‘ (∗) 2 ‘ {a..b} = {2∗a..2∗b+1} (is ?l = ?r)
for a b
  proof
    show ?l ⊆ ?r by auto
  next
     have ∃ x2∈{a..b}. x ∈ {Suc (2∗x2), 2∗x2} if ∗: x ∈ {2∗a .. 2∗b+1}
for x
    proof −
      have x div 2 ∈ {a..b} using ∗ by auto
      moreover have x ∈ {2 ∗ (x div 2), Suc(2 ∗ (x div 2))} by auto
      ultimately show ?thesis by blast
    qed
    thus ?r ⊆ ?l by fastforce
  qed
  case (Node l x r)
  hence size l = size r ∨ size l = size r + 1 (is ?A ∨ ?B) by auto
  thus ?case
  proof
    assume ?A
    with Node show ?thesis by (auto simp: ∗)
```

```
  next
    assume ?B
    with Node show ?thesis by (auto simp: ∗ atLeastAtMostSuc_conv)
  qed
qed
```

The other direction is more complicated. The following proof is due to Thomas Sewell.

```
lemma disj_evens_odds: (∗) 2 ‘ A ∩ Suc ‘ (∗) 2 ‘ B = {}
using double_not_eq_Suc_double by auto

lemma card_braun_indices: card (braun_indices t) = size t
proof (induction t)
  case Leaf thus ?case by simp
next
  case Node
  thus ?case
   by(auto simp: UNION_singleton_eq_range finite_braun_indices card_Un_disjoint
                    card_insert_if disj_evens_odds card_image inj_on_def
braun_indices1 )
qed

lemma braun_indices_intvl_base_1 :
  assumes bi: braun_indices t = {m..n}
  shows {m..n} = {1 ..size t}
proof (cases t = Leaf )
  case True then show ?thesis using bi by simp
next
  case False
  note eqs = eqset_imp_iff [OF bi]
  from eqs[of 0] have 0: 0 < m
    by (simp add: braun_indices1 )
  from eqs[of 1] have 1: m ≤ 1
    by (cases t; simp add: False)
  from 0 1 have eq1: m = 1 by simp
  from card_braun_indices[of t] show ?thesis
    by (simp add: bi eq1 )
qed

lemma even_of_intvl_intvl:
  fixes S :: nat set
  assumes S = {m..n} ∩ {i. even i}
  shows ∃ m′ n′. S = (λi. i ∗ 2) ‘ {m′..n′}
  apply (rule exI [where x=Suc m div 2], rule exI [where x=n div 2])
```

```
  apply (fastforce simp add: assms mult.commute)
  done

lemma odd_of_intvl_intvl:
  fixes S :: nat set
  assumes S = {m..n} ∩ {i. odd i}
  shows ∃ m′ n′. S = Suc ‘ (λi. i ∗ 2) ‘ {m′..n′}
proof −
  have step1: ∃ m′. S = Suc ‘ ({m′..n − 1} ∩ {i. even i})
    apply (rule_tac x=if n = 0 then 1 else m − 1 in exI)
    apply (auto simp: assms image_def elim!: oddE)
    done
  thus ?thesis
    by (metis even_of_intvl_intvl)
qed

lemma image_int_eq_image:
  (∀ i ∈ S. f i ∈ T) =⇒ (f ‘ S) ∩ T = f ‘ S
  (∀ i ∈ S. f i ∉ T) =⇒ (f ‘ S) ∩ T = {}
  by auto

lemma braun_indices1_le:
  i ∈ braun_indices t =⇒ Suc 0 ≤ i
  using braun_indices1 not_less_eq_eq by blast

lemma braun_if_braun_indices: braun_indices t = {1..size t} =⇒ braun
t
proof(induction t)
case Leaf
  then show ?case by simp
next
  case (Node l x r)
  obtain t where t: t = Node l x r by simp
  from Node.prems have eq: {2 .. size t} = (λi. i ∗ 2) ‘ braun_indices l
∪ Suc ‘ (λi. i ∗ 2) ‘ braun_indices r
    (is ?R = ?S ∪ ?T)
    apply clarsimp
    apply (drule_tac f=λS. S ∩ {2..} in arg_cong)
   apply (simp add: t mult.commute Int_Un_distrib2 image_int_eq_image
braun_indices1_le)
    done
  then have ST: ?S = ?R ∩ {i. even i} ?T = ?R ∩ {i. odd i}
    by (simp_all add: Int_Un_distrib2 image_int_eq_image)
  from ST have l: braun_indices l = {1 .. size l}
```

by (*fastforce dest*: *braun_indices_intvl_base_1 dest*!: *even_of_intvl_intvl*
simp: *mult.commute inj_image_eq_iff*[*OF inj_onI*])
from *ST* **have** *r*: *braun_indices r* = {*1* .. *size r*}
by (*fastforce dest*: *braun_indices_intvl_base_1 dest*!: *odd_of_intvl_intvl*
simp: *mult.commute inj_image_eq_iff*[*OF inj_onI*])
note *STa* = *ST*[*THEN eqset_imp_iff*, *THEN iffD2*]
note *STb* = *STa*[*of size t*] *STa*[*of size t* − *1*]
then have *sizes*: *size l* = *size r* ∨ *size l* = *size r* + *1*
apply (*clarsimp simp*: *t l r inj_image_mem_iff*[*OF inj_onI*])
apply (*cases even* (*size l*); *cases even* (*size r*); *clarsimp elim*!: *oddE*; *fastforce*)
done
from *l r sizes* **show** *?case*
by (*clarsimp simp*: *Node.IH*)
qed

lemma *braun_iff_braun_indices*: *braun t* ⟷ *braun_indices t* = {*1*..*size t*}
using *braun_if_braun_indices braun_indices_if_braun* **by** *blast*

end

39 Arrays via Braun Trees

theory *Array_Braun*
imports
Array_Specs
Braun_Tree
begin

39.1 Array

fun *lookup1* :: *'a tree* ⇒ *nat* ⇒ *'a* **where**
lookup1 (*Node l x r*) *n* = (*if n=1 then x else lookup1* (*if even n then l else r*) (*n div 2*))

fun *update1* :: *nat* ⇒ *'a* ⇒ *'a tree* ⇒ *'a tree* **where**
update1 n x Leaf = *Node Leaf x Leaf* |
update1 n x (*Node l a r*) =
(*if n=1 then Node l x r else*
if even n then Node (*update1* (*n div 2*) *x l*) *a r*
else Node l a (*update1* (*n div 2*) *x r*))

fun *adds* :: *'a list* ⇒ *nat* ⇒ *'a tree* ⇒ *'a tree* **where**
adds [] *n t* = *t* |
adds (*x*#*xs*) *n t* = *adds xs* (*n*+*1*) (*update1* (*n*+*1*) *x t*)

fun *list* :: *'a tree* ⇒ *'a list* **where**
list Leaf = [] |
list (*Node l x r*) = *x* # *splice* (*list l*) (*list r*)

39.1.1 Functional Correctness

lemma *size_list*: *size*(*list t*) = *size t*
by(*induction t*)(*auto*)

lemma *minus1_div2*: (*n* − *Suc 0*) *div 2* = (*if odd n then n div 2 else n div 2* − *1*)
by *auto arith*

lemma *nth_splice*: ⟦ *n* < *size xs* + *size ys*; *size ys* ≤ *size xs*; *size xs* ≤ *size ys* + *1* ⟧
⟹ *splice xs ys* ! *n* = (*if even n then xs else ys*) ! (*n div 2*)
apply(*induction xs ys arbitrary*: *n rule*: *splice.induct*)
apply (*auto simp*: *nth_Cons' minus1_div2*)
done

lemma *div2_in_bounds*:
⟦ *braun* (*Node l x r*); *n* ∈ {*1*..*size*(*Node l x r*)}; *n* > *1* ⟧ ⟹
(*odd n* ⟶ *n div 2* ∈ {*1*..*size r*}) ∧ (*even n* ⟶ *n div 2* ∈ {*1*..*size l*})
by *auto arith*

declare *upt_Suc*[*simp del*]

lookup1 **lemma** *nth_list_lookup1*: ⟦*braun t*; *i* < *size t*⟧ ⟹ *list t* ! *i* = *lookup1 t* (*i*+*1*)
proof(*induction t arbitrary*: *i*)
case *Leaf* **thus** *?case* **by** *simp*
next
case *Node*
thus *?case* **using** *div2_in_bounds*[*OF Node.prems*(*1*), *of i*+*1*]
by (*auto simp*: *nth_splice minus1_div2 size_list*)
qed

lemma *list_eq_map_lookup1*: *braun t* ⟹ *list t* = *map* (*lookup1 t*) [*1*..<*size t* + *1*]

by(*auto simp add*: *list_eq_iff_nth_eq size_list nth_list_lookup1*)

update1 **lemma** *size_update1*: ⟦ *braun t*; *n* ∈ {*1*.. *size t*} ⟧ ⟹ *size*(*update1 n x t*) = *size t*
proof(*induction t arbitrary*: *n*)
  case *Leaf* **thus** *?case* **by** *simp*
next
  case *Node* **thus** *?case* **using** *div2_in_bounds*[*OF Node.prems*] **by** *simp*
qed

lemma *braun_update1*: ⟦*braun t*; *n* ∈ {*1*.. *size t*} ⟧ ⟹ *braun*(*update1 n x t*)
proof(*induction t arbitrary*: *n*)
  case *Leaf* **thus** *?case* **by** *simp*
next
  case *Node* **thus** *?case*
    using *div2_in_bounds*[*OF Node.prems*] **by** (*simp add*: *size_update1*)
qed

lemma *lookup1_update1*: ⟦ *braun t*; *n* ∈ {*1*.. *size t*} ⟧ ⟹
  lookup1 (*update1 n x t*) *m* = (*if n*=*m then x else lookup1 t m*)
proof(*induction t arbitrary*: *m n*)
  case *Leaf*
  then show *?case* **by** *simp*
next
  have *aux*: ⟦ *odd n*; *odd m* ⟧ ⟹ *n div 2* = (*m*::*nat*) *div 2* ⟷ *m*=*n* **for** *m n*
    using *odd_two_times_div_two_succ* **by** *fastforce*
  case *Node*
  thus *?case* **using** *div2_in_bounds*[*OF Node.prems*] **by** (*auto simp*: *aux*)
qed

lemma *list_update1*: ⟦ *braun t*; *n* ∈ {*1*.. *size t*} ⟧ ⟹ *list*(*update1 n x t*) = (*list t*)[*n*−*1* := *x*]
by(*auto simp add*: *list_eq_map_lookup1 list_eq_iff_nth_eq lookup1_update1 size_update1 braun_update1*)

A second proof of ⟦*braun ?t*; *?n* ∈ {*1*..*size ?t*}⟧ ⟹ *list* (*update1 ?n ?x ?t*) = (*list ?t*)[*?n* − *1* := *?x*]:

lemma *diff1_eq_iff*: *n* > *0* ⟹ *n* − *Suc 0* = *m* ⟷ *n* = *m*+*1*
by *arith*

lemma *list_update_splice*:
  ⟦ *n* < *size xs* + *size ys*; *size ys* ≤ *size xs*; *size xs* ≤ *size ys* + *1* ⟧ ⟹

(*splice xs ys*) [*n* := *x*] =
(*if even n then splice* (*xs*[*n div 2* := *x*]) *ys else splice xs* (*ys*[*n div 2* := *x*]))
by(*induction xs ys arbitrary*: *n rule*: *splice.induct*) (*auto split*: *nat.split*)

lemma *list_update2*: ⟦ *braun t*; *n* ∈ {*1*.. *size t*} ⟧ ⟹ *list*(*update1 n x t*) = (*list t*)[*n*−*1* := *x*]
proof(*induction t arbitrary*: *n*)
case *Leaf* **thus** *?case* **by** *simp*
next
case (*Node l a r*) **thus** *?case* **using** *div2_in_bounds*[*OF Node.prems*]
by(*auto simp*: *list_update_splice diff1_eq_iff size_list split*: *nat.split*)
qed

adds **lemma** *splice_last*: **shows**
size ys ≤ *size xs* ⟹ *splice* (*xs* @ [*x*]) *ys* = *splice xs ys* @ [*x*]
and *size ys*+*1* ≥ *size xs* ⟹ *splice xs* (*ys* @ [*y*]) = *splice xs ys* @ [*y*]
by(*induction xs ys arbitrary*: *x y rule*: *splice.induct*) (*auto*)

lemma *list_add_hi*: *braun t* ⟹ *list*(*update1* (*Suc*(*size t*)) *x t*) = *list t* @ [*x*]
by(*induction t*)(*auto simp*: *splice_last size_list*)

lemma *size_add_hi*: *braun t* ⟹ *m* = *size t* ⟹ *size*(*update1* (*Suc m*) *x t*) = *size t* + *1*
by(*induction t arbitrary*: *m*)(*auto*)

lemma *braun_add_hi*: *braun t* ⟹ *braun*(*update1* (*Suc*(*size t*)) *x t*)
by(*induction t*)(*auto simp*: *size_add_hi*)

lemma *size_braun_adds*:
⟦ *braun t*; *size t* = *n* ⟧ ⟹ *size*(*adds xs n t*) = *size t* + *length xs* ∧ *braun* (*adds xs n t*)
by(*induction xs arbitrary*: *t n*)(*auto simp*: *braun_add_hi size_add_hi*)

lemma *list_adds*: ⟦ *braun t*; *size t* = *n* ⟧ ⟹ *list*(*adds xs n t*) = *list t* @ *xs*
by(*induction xs arbitrary*: *t n*)(*auto simp*: *size_braun_adds list_add_hi size_add_hi braun_add_hi*)

39.1.2 Array Implementation

interpretation *A*: *Array*
where *lookup* = λ(*t*,*l*) *n*. *lookup1 t* (*n*+*1*)
and *update* = λ*n x* (*t*,*l*). (*update1* (*n*+*1*) *x t*, *l*)
and *len* = λ(*t*,*l*). *l*

and *array* = λ*xs*. (*adds xs 0 Leaf*, *length xs*)
and *invar* = λ(*t*,*l*). *braun t* ∧ *l* = *size t*
and *list* = λ(*t*,*l*). *list t*
proof (*standard*, *goal_cases*)
  case *1* **thus** *?case* **by** (*simp add*: *nth_list_lookup1 split*: *prod.splits*)
next
  case *2* **thus** *?case* **by** (*simp add*: *list_update1 split*: *prod.splits*)
next
  case *3* **thus** *?case* **by** (*simp add*: *size_list split*: *prod.splits*)
next
  case *4* **thus** *?case* **by** (*simp add*: *list_adds*)
next
  case *5* **thus** *?case* **by** (*simp add*: *braun_update1 size_update1 split*: *prod.splits*)
next
  case *6* **thus** *?case* **by** (*simp add*: *size_braun_adds split*: *prod.splits*)
qed

39.2 Flexible Array

fun *add_lo* **where**
add_lo x Leaf = *Node Leaf x Leaf* |
add_lo x (*Node l a r*) = *Node* (*add_lo a r*) *x l*

fun *merge* **where**
merge Leaf r = *r* |
merge (*Node l a r*) *rr* = *Node rr a* (*merge l r*)

fun *del_lo* **where**
del_lo Leaf = *Leaf* |
del_lo (*Node l a r*) = *merge l r*

fun *del_hi* :: *nat* ⇒ *'a tree* ⇒ *'a tree* **where**
del_hi n Leaf = *Leaf* |
del_hi n (*Node l x r*) =
  (*if n* = *1 then Leaf*
   else if even n
      then Node (*del_hi* (*n div 2*) *l*) *x r*
      else Node l x (*del_hi* (*n div 2*) *r*))

39.2.1 Functional Correctness

add_lo **lemma** *list_add_lo*: *braun t* ⟹ *list* (*add_lo a t*) = *a* # *list t*
by(*induction t arbitrary*: *a*) *auto*

lemma *braun_add_lo*: *braun t* $\Longrightarrow$ *braun*(*add_lo x t*)
by(*induction t arbitrary*: *x*) (*auto simp add*: *list_add_lo simp flip*: *size_list*)

del_lo **lemma** *list_merge*: *braun* (*Node l x r*) $\Longrightarrow$ *list*(*merge l r*) = *splice* (*list l*) (*list r*)
by (*induction l r rule*: *merge.induct*) *auto*

lemma *braun_merge*: *braun* (*Node l x r*) $\Longrightarrow$ *braun*(*merge l r*)
by (*induction l r rule*: *merge.induct*)(*auto simp add*: *list_merge simp flip*: *size_list*)

lemma *list_del_lo*: *braun t* $\Longrightarrow$ *list*(*del_lo t*) = *tl* (*list t*)
by (*cases t*) (*simp_all add*: *list_merge*)

lemma *braun_del_lo*: *braun t* $\Longrightarrow$ *braun*(*del_lo t*)
by (*cases t*) (*simp_all add*: *braun_merge*)

del_hi **lemma** *list_Nil_iff*: *list t* = [] $\longleftrightarrow$ *t* = *Leaf*
by(*cases t*) *simp_all*

lemma *butlast_splice*: *butlast* (*splice xs ys*) =
(*if size xs* > *size ys then splice* (*butlast xs*) *ys else splice xs* (*butlast ys*))
by(*induction xs ys rule*: *splice.induct*) (*auto*)

lemma *list_del_hi*: *braun t* $\Longrightarrow$ *size t* = *st* $\Longrightarrow$ *list*(*del_hi st t*) = *butlast*(*list t*)
apply(*induction t arbitrary*: *st*)
by(*auto simp*: *list_Nil_iff size_list butlast_splice*)

lemma *braun_del_hi*: *braun t* $\Longrightarrow$ *size t* = *st* $\Longrightarrow$ *braun*(*del_hi st t*)
apply(*induction t arbitrary*: *st*)
by(*auto simp*: *list_del_hi simp flip*: *size_list*)

39.2.2 Flexible Array Implementation

interpretation *AF*: *Array_Flex*
where *lookup* = λ(*t*,*l*) *n*. *lookup1 t* (*n*+*1*)
and *update* = λ*n x* (*t*,*l*). (*update1* (*n*+*1*) *x t*, *l*)
and *len* = λ(*t*,*l*). *l*
and *array* = λ*xs*. (*adds xs 0 Leaf*, *length xs*)
and *invar* = λ(*t*,*l*). *braun t* $\wedge$ *l* = *size t*
and *list* = λ(*t*,*l*). *list t*
and *add_lo* = λ*x* (*t*,*l*). (*add_lo x t*, *l*+*1*)

and *del_lo* = λ(*t*,*l*). (*del_lo t*, *l*−*1*)
and *add_hi* = λ*x* (*t*,*l*). (*update1* (*Suc l*) *x t*, *l*+*1*)
and *del_hi* = λ(*t*,*l*). (*del_hi l t*, *l*−*1*)
proof (*standard*, *goal_cases*)
case *1* **thus** *?case* **by** (*simp add*: *list_add_lo split*: *prod.splits*)
next
case *2* **thus** *?case* **by** (*simp add*: *list_del_lo split*: *prod.splits*)
next
case *3* **thus** *?case* **by** (*simp add*: *list_add_hi braun_add_hi split*: *prod.splits*)
next
case *4* **thus** *?case* **by** (*simp add*: *list_del_hi split*: *prod.splits*)
next
case *5* **thus** *?case* **by** (*simp add*: *braun_add_lo list_add_lo flip*: *size_list split*: *prod.splits*)
next
case *6* **thus** *?case* **by** (*simp add*: *braun_del_lo list_del_lo flip*: *size_list split*: *prod.splits*)
next
case *7* **thus** *?case* **by** (*simp add*: *size_add_hi braun_add_hi split*: *prod.splits*)
next
case *8* **thus** *?case* **by** (*simp add*: *braun_del_hi list_del_hi flip*: *size_list split*: *prod.splits*)
qed

39.3 Faster

39.3.1 Size

fun *diff* :: ′*a tree* ⇒ *nat* ⇒ *nat* **where**
diff Leaf _ = *0* |
diff (*Node l x r*) *n* = (*if n*=*0 then 1 else if even n then diff r* (*n div 2* − *1*) *else diff l* (*n div 2*))

fun *size_fast* :: ′*a tree* ⇒ *nat* **where**
size_fast Leaf = *0* |
size_fast (*Node l x r*) = (*let n* = *size_fast r in 1* + *2*∗*n* + *diff l n*)

declare *Let_def*[*simp*]

lemma *diff*: *braun t* ⟹ *size t* : {*n*, *n* + *1*} ⟹ *diff t n* = *size t* − *n*
by(*induction t arbitrary*: *n*) *auto*

lemma *size_fast*: *braun t* ⟹ *size_fast t* = *size t*
by(*induction t*) (*auto simp add*: *diff*)

39.3.2 Initialization with 1 element

fun *braun_of_naive* :: *'a* ⇒ *nat* ⇒ *'a tree* **where**
braun_of_naive x n = (*if n=0 then Leaf*
else let m = (*n−1*) *div 2*
in if odd n then Node (*braun_of_naive x m*) *x* (*braun_of_naive x m*)
else Node (*braun_of_naive x* (*m* + *1*)) *x* (*braun_of_naive x m*))

fun *braun2_of* :: *'a* ⇒ *nat* ⇒ *'a tree* ∗ *'a tree* **where**
braun2_of x n = (*if n* = *0 then* (*Leaf*, *Node Leaf x Leaf*)
else let (*s*,*t*) = *braun2_of x* ((*n−1*) *div 2*)
in if odd n then (*Node s x s*, *Node t x s*) *else* (*Node t x s*, *Node t x t*))

definition *braun_of* :: *'a* ⇒ *nat* ⇒ *'a tree* **where**
braun_of x n = *fst* (*braun2_of x n*)

declare *braun2_of.simps* [*simp del*]

lemma *braun2_of_size_braun*: *braun2_of x n* = (*s*,*t*) ⟹ *size s* = *n* ∧ *size t* = *n+1* ∧ *braun s* ∧ *braun t*
proof(*induction x n arbitrary*: *s t rule*: *braun2_of.induct*)
case (*1 x n*)
then show *?case*
by (*auto simp*: *braun2_of.simps*[*of x n*] *split*: *prod.splits if_splits*) *presburger+*
qed

lemma *braun2_of_replicate*:
braun2_of x n = (*s*,*t*) ⟹ *list s* = *replicate n x* ∧ *list t* = *replicate* (*n+1*) *x*
proof(*induction x n arbitrary*: *s t rule*: *braun2_of.induct*)
case (*1 x n*)
have *x* # *replicate m x* = *replicate* (*m+1*) *x* **for** *m* **by** *simp*
with *1* **show** *?case*
apply (*auto simp*: *braun2_of.simps*[*of x n*] *replicate.simps*(*2*)[*of 0 x*]
simp del: *replicate.simps*(*2*) *split*: *prod.splits if_splits*)
by *presburger+*
qed

corollary *braun_braun_of*: *braun*(*braun_of x n*)
unfolding *braun_of_def* **by** (*metis eq_fst_iff braun2_of_size_braun*)

corollary *list_braun_of*: *list*(*braun_of x n*) = *replicate n x*
unfolding *braun_of_def* **by** (*metis eq_fst_iff braun2_of_replicate*)

39.3.3 Proof Infrastructure

Originally due to Thomas Sewell.

take_nths **fun** *take_nths* :: *nat* ⇒ *nat* ⇒ *'a list* ⇒ *'a list* **where**
take_nths i k [] = [] |
take_nths i k (*x* # *xs*) = (*if i* = *0 then x* # *take_nths* (*2^k* − *1*) *k xs*
else take_nths (*i* − *1*) *k xs*)

This is the more concise definition but seems to complicate the proofs:

```
lemma take_nths_eq_nths: take_nths i k xs = nths xs (⋃n. {n*2^k + i})
proof(induction xs arbitrary: i)
  case Nil
  then show ?case by simp
next
  case (Cons x xs)
  show ?case
  proof cases
    assume [simp]: i = 0
    have (⋃n. {(n+1) * 2 ^ k − 1}) = {m. ∃n. Suc m = n * 2 ^ k}
      apply (auto simp del: mult_Suc)
      by (metis diff_Suc_Suc diff_zero mult_eq_0_iff not0_implies_Suc)
    thus ?thesis by (simp add: Cons.IH ac_simps nths_Cons)
  next
    assume [arith]: i ≠ 0
    have (⋃n. {n * 2 ^ k + i − 1}) = {m. ∃n. Suc m = n * 2 ^ k + i}
      apply auto
      by (metis diff_Suc_Suc diff_zero)
    thus ?thesis by (simp add: Cons.IH nths_Cons)
  qed
qed

lemma take_nths_drop:
  take_nths i k (drop j xs) = take_nths (i + j) k xs
by (induct xs arbitrary: i j; simp add: drop_Cons split: nat.split)

lemma take_nths_00:
  take_nths 0 0 xs = xs
by (induct xs; simp)

lemma splice_take_nths:
  splice (take_nths 0 (Suc 0) xs) (take_nths (Suc 0) (Suc 0) xs) = xs
by (induct xs; simp)
```

```
lemma take_nths_take_nths:
  take_nths i m (take_nths j n xs) = take_nths ((i * 2^n) + j) (m + n) xs
by (induct xs arbitrary: i j; simp add: algebra_simps power_add)

lemma take_nths_empty:
  (take_nths i k xs = []) = (length xs ≤ i)
by (induction xs arbitrary: i k) auto

lemma hd_take_nths:
  i < length xs ⟹ hd(take_nths i k xs) = xs ! i
by (induction xs arbitrary: i k) auto

lemma take_nths_01_splice:
  ⟦ length xs = length ys ∨ length xs = length ys + 1 ⟧ ⟹
   take_nths 0 (Suc 0) (splice xs ys) = xs ∧
   take_nths (Suc 0) (Suc 0) (splice xs ys) = ys
by (induct xs arbitrary: ys; case_tac ys; simp)

lemma length_take_nths_00:
  length (take_nths 0 (Suc 0) xs) = length (take_nths (Suc 0) (Suc 0) xs)
∨
   length (take_nths 0 (Suc 0) xs) = length (take_nths (Suc 0) (Suc 0) xs)
+ 1
by (induct xs) auto

braun_list   fun braun_list :: 'a tree ⇒ 'a list ⇒ bool where
braun_list Leaf xs = (xs = []) |
braun_list (Node l x r) xs = (xs ≠ [] ∧ x = hd xs ∧
    braun_list l (take_nths 1 1 xs) ∧
    braun_list r (take_nths 2 1 xs))

lemma braun_list_eq:
  braun_list t xs = (braun t ∧ xs = list t)
proof (induct t arbitrary: xs)
  case Leaf
  show ?case by simp
next
  case Node
  show ?case
    using length_take_nths_00[of xs] splice_take_nths[of xs]
   by (auto simp: neq_Nil_conv Node.hyps size_list[symmetric] take_nths_01_splice)
qed
```

39.3.4 Converting a list of elements into a Braun tree

fun *nodes* :: *'a tree list* ⇒ *'a list* ⇒ *'a tree list* ⇒ *'a tree list* **where**
nodes (*l#ls*) (*x#xs*) (*r#rs*) = *Node l x r* # *nodes ls xs rs* |
nodes (*l#ls*) (*x#xs*) [] = *Node l x Leaf* # *nodes ls xs* [] |
nodes [] (*x#xs*) (*r#rs*) = *Node Leaf x r* # *nodes* [] *xs rs* |
nodes [] (*x#xs*) [] = *Node Leaf x Leaf* # *nodes* [] *xs* [] |
nodes ls [] *rs* = []

fun *brauns* :: *nat* ⇒ *'a list* ⇒ *'a tree list* **where**
brauns k xs = (*if xs* = [] *then* [] *else*
  let ys = *take* (*2^k*) *xs*;
    zs = *drop* (*2^k*) *xs*;
    ts = *brauns* (*k+1*) *zs*
  in nodes ts ys (*drop* (*2^k*) *ts*))

declare *brauns.simps*[*simp del*]

definition *brauns1* :: *'a list* ⇒ *'a tree* **where**
brauns1 xs = (*if xs* = [] *then Leaf else brauns 0 xs* ! *0*)

fun *T_brauns* :: *nat* ⇒ *'a list* ⇒ *nat* **where**
T_brauns k xs = (*if xs* = [] *then 0 else*
  let ys = *take* (*2^k*) *xs*;
    zs = *drop* (*2^k*) *xs*;
    ts = *brauns* (*k+1*) *zs*
  in 4 ∗ *min* (*2^k*) (*length xs*) + *T_brauns* (*k+1*) *zs*)

Functional correctness The proof is originally due to Thomas Sewell.

lemma *length_nodes*:
 length (*nodes ls xs rs*) = *length xs*
by (*induct ls xs rs rule*: *nodes.induct*; *simp*)

lemma *nth_nodes*:
 i < *length xs* ⟹ *nodes ls xs rs* ! *i* =
 Node (*if i* < *length ls then ls* ! *i else Leaf*) (*xs* ! *i*)
  (*if i* < *length rs then rs* ! *i else Leaf*)
by (*induct ls xs rs arbitrary*: *i rule*: *nodes.induct*;
  simp add: *nth_Cons split*: *nat.split*)

theorem *length_brauns*:
 length (*brauns k xs*) = *min* (*length xs*) (*2 ^ k*)
proof (*induct xs arbitrary*: *k rule*: *measure_induct_rule*[**where** *f*=*length*])
 case (*less xs*) **thus** *?case* **by** (*simp add*: *brauns.simps*[*of k xs*] *length_nodes*)

qed

theorem *braums_correct*:
i < *min* (*length xs*) (*2* ^ *k*) ⟹ *braun_list* (*brauns k xs* ! *i*) (*take_nths i k xs*)
proof (*induct xs arbitrary*: *i k rule*: *measure_induct_rule*[**where** *f*=*length*])
 case (*less xs*)
 have *xs* ≠ [] **using** *less.prems* **by** *auto*
 let *?zs* = *drop* (*2*^*k*) *xs*
 let *?ts* = *brauns* (*Suc k*) *?zs*
 from *less.hyps*[*of ?zs _ Suc k*]
 have *IH*: ⟦ *j* = *i* + *2* ^ *k*; *i* < *min* (*length ?zs*) (*2* ^ (*k*+*1*)) ⟧ ⟹
  braun_list (*?ts* ! *i*) (*take_nths j* (*Suc k*) *xs*) **for** *i j*
  using ‹*xs* ≠ []› **by** (*simp add*: *take_nths_drop*)
 show *?case*
  using *less.prems*
  by (*auto simp*: *brauns.simps*[*of k xs*] *nth_nodes take_nths_take_nths*
    IH take_nths_empty hd_take_nths length_brauns)
qed

corollary *brauns1_correct*:
 braun (*brauns1 xs*) ∧ *list* (*brauns1 xs*) = *xs*
using *brauns_correct*[*of 0 xs 0*]
by (*simp add*: *brauns1_def braun_list_eq take_nths_00*)

Running Time Analysis **theorem** *T_brauns*:
 T_brauns k xs = *4* ∗ *length xs*
proof (*induction xs arbitrary*: *k rule*: *measure_induct_rule*[**where** *f* = *length*])
 case (*less xs*)
 show *?case*
 proof *cases*
  assume *xs* = []
  thus *?thesis* **by**(*simp*)
 next
  assume *xs* ≠ []
  let *?zs* = *drop* (*2*^*k*) *xs*
  have *T_brauns k xs* = *T_brauns* (*k*+*1*) *?zs* + *4* ∗ *min* (*2*^*k*) (*length xs*)
   using ‹*xs* ≠ []› **by**(*simp*)
  also have . . . = *4* ∗ *length ?zs* + *4* ∗ *min* (*2*^*k*) (*length xs*)
   using *less*[*of ?zs k*+*1*] ‹*xs* ≠ []›
   by (*simp*)

also have ... = *4* ∗ *length xs*
by(*simp*)
finally show *?case* **.**
qed
qed

39.3.5 Converting a Braun Tree into a List of Elements

The code and the proof are originally due to Thomas Sewell (except running time).

function *list_fast_rec* :: *'a tree list* ⇒ *'a list* **where**
list_fast_rec ts = (*let us* = *filter* (λ*t*. *t* ≠ *Leaf*) *ts in*
if us = [] *then* [] *else*
map value us @ *list_fast_rec* (*map left us* @ *map right us*))
by (*pat_completeness*, *auto*)

lemma *list_fast_rec_term1*: *ts* ≠ [] ⟹ *Leaf* ∉ *set ts* ⟹
sum_list (*map* (*size o left*) *ts*) + *sum_list* (*map* (*size o right*) *ts*) < *sum_list* (*map size ts*)
apply (*clarsimp simp*: *sum_list_addf*[*symmetric*] *sum_list_map_filter'*)
apply (*rule sum_list_strict_mono*; *clarsimp?*)
apply (*case_tac x*; *simp*)
done

lemma *list_fast_rec_term*: *us* ≠ [] ⟹ *us* = *filter* (λ*t*. *t* ≠ ⟨⟩) *ts* ⟹
sum_list (*map* (*size o left*) *us*) + *sum_list* (*map* (*size o right*) *us*) < *sum_list* (*map size ts*)
apply (*rule order_less_le_trans*, *rule list_fast_rec_term1*, *simp_all*)
apply (*rule sum_list_filter_le_nat*)
done

termination
apply (*relation measure* (*sum_list o map size*))
apply *simp*
apply (*simp add*: *list_fast_rec_term*)
done

declare *list_fast_rec.simps*[*simp del*]

definition *list_fast* :: *'a tree* ⇒ *'a list* **where**
list_fast t = *list_fast_rec* [*t*]

function *T_list_fast_rec* :: *'a tree list* ⇒ *nat* **where**
T_list_fast_rec ts = (*let us* = *filter* (λ*t*. *t* ≠ *Leaf*) *ts*

in length ts + (*if us* = [] *then 0 else*
5 ∗ *length us* + *T_list_fast_rec* (*map left us* @ *map right us*)))
by (*pat_completeness*, *auto*)

termination
apply (*relation measure* (*sum_list o map size*))
apply *simp*
apply (*simp add*: *list_fast_rec_term*)
done

declare *T_list_fast_rec.simps*[*simp del*]

Functional Correctness **lemma** *list_fast_rec_all_Leaf*:
∀ *t* ∈ *set ts*. *t* = *Leaf* ⟹ *list_fast_rec ts* = []
by (*simp add*: *filter_empty_conv list_fast_rec.simps*)

lemma *take_nths_eq_single*:
length xs − *i* < *2*^*n* ⟹ *take_nths i n xs* = *take 1* (*drop i xs*)
by (*induction xs arbitrary*: *i n*; *simp add*: *drop_Cons′*)

lemma *braun_list_Nil*:
braun_list t [] = (*t* = *Leaf*)
by (*cases t*; *simp*)

lemma *braun_list_not_Nil*: *xs* ≠ [] ⟹
braun_list t xs =
(∃ *l x r*. *t* = *Node l x r* ∧ *x* = *hd xs* ∧
braun_list l (*take_nths 1 1 xs*) ∧
braun_list r (*take_nths 2 1 xs*))
by(*cases t*; *simp*)

theorem *list_fast_rec_correct*:
⟦ *length ts* = *2* ^ *k*; ∀ *i* < *2* ^ *k*. *braun_list* (*ts* ! *i*) (*take_nths i k xs*) ⟧
⟹ *list_fast_rec ts* = *xs*
proof (*induct xs arbitrary*: *k ts rule*: *measure_induct_rule*[**where** *f*=*length*])
case (*less xs*)
show *?case*
proof (*cases length xs* < *2* ^ *k*)
case *True*
from *less.prems True* **have** *filter*:
∃ *n*. *ts* = *map* (λ*x*. *Node Leaf x Leaf*) *xs* @ *replicate n Leaf*
apply (*rule_tac x*=*length ts* − *length xs* **in** *exI*)
apply (*clarsimp simp*: *list_eq_iff_nth_eq*)

apply(*auto simp*: *nth_append braun_list_not_Nil take_nths_eq_single braun_list_Nil hd_drop_conv_nth*)
  done
 thus *?thesis*
 by (*clarsimp simp*: *list_fast_rec.simps*[*of ts*] *o_def list_fast_rec_all_Leaf*)
next
 case *False*
 with *less.prems*(*2*) **have** ∗:
 $\forall i < 2 \mathbin{\hat{}} k.\ ts\ !\ i \neq Leaf$
 $\wedge\ value\ (ts\ !\ i) = xs\ !\ i$
 $\wedge\ braun\_list\ (left\ (ts\ !\ i))\ (take\_nths\ (i + 2 \mathbin{\hat{}} k)\ (Suc\ k)\ xs)$
 $\wedge\ (\forall ys.\ ys = take\_nths\ (i + 2 * 2 \mathbin{\hat{}} k)\ (Suc\ k)\ xs$
 $\longrightarrow braun\_list\ (right\ (ts\ !\ i))\ ys)$
 by (*auto simp*: *take_nths_empty hd_take_nths braun_list_not_Nil take_nths_take_nths*
 algebra_simps)
 have *1*: *map value ts* = *take* (*2* ^ *k*) *xs*
 using *less.prems*(*1*) *False* **by** (*simp add*: *list_eq_iff_nth_eq* ∗)
 have *2*: *list_fast_rec* (*map left ts* @ *map right ts*) = *drop* (*2* ^ *k*) *xs*
 using *less.prems*(*1*) *False*
 by (*auto intro*!: *Nat.diff_less less.hyps*[**where** *k*= *Suc k*]
 simp: *nth_append* ∗ *take_nths_drop algebra_simps*)
 from *less.prems*(*1*) *False* **show** *?thesis*
 by (*auto simp*: *list_fast_rec.simps*[*of ts*] *1 2* ∗ *all_set_conv_all_nth*)
qed
qed

corollary *list_fast_correct*:
 braun t ⟹ *list_fast t* = *list t*
by (*simp add*: *list_fast_def take_nths_00 braun_list_eq list_fast_rec_correct*[**where** *k*=*0*])

Running Time Analysis **lemma** *sum_tree_list_children*: ∀ *t* ∈ *set ts*. *t* ≠ *Leaf* ⟹
 $(\sum t{\leftarrow}ts.\ k * size\ t) = (\sum t \leftarrow map\ left\ ts\ @\ map\ right\ ts.\ k * size\ t) + k * length\ ts$
by(*induction ts*)(*auto simp add*: *neq_Leaf_iff algebra_simps*)

theorem *T_list_fast_rec_ub*:
 $T\_list\_fast\_rec\ ts \le sum\_list\ (map\ (\lambda t.\ 7{*}size\ t + 1)\ ts)$
proof (*induction ts rule*: *measure_induct_rule*[**where** *f*=*sum_list o map size*])
 case (*less ts*)

let *?us* = *filter* (λ*t*. *t* ≠ *Leaf*) *ts*
show *?case*
proof *cases*
assume *?us* = []
thus *?thesis* **using** *T_list_fast_rec.simps*[*of ts*]
by(*simp add*: *sum_list_Suc*)
next
assume *?us* ≠ []
let *?children* = *map left ?us* @ *map right ?us*
have *T_list_fast_rec ts* = *T_list_fast_rec ?children* + *5* ∗ *length ?us* + *length ts*
using ‹*?us* ≠ []› *T_list_fast_rec.simps*[*of ts*] **by**(*simp*)
also have . . . ≤ (∑ *t*←*?children*. *7* ∗ *size t* + *1*) + *5* ∗ *length ?us* + *length ts*
using *less*[*of ?children*] *list_fast_rec_term*[*of ?us*] ‹*?us* ≠ []›
by (*simp*)
also have . . . = (∑ *t*←*?children*. *7*∗*size t*) + *7* ∗ *length ?us* + *length ts*
by(*simp add*: *sum_list_Suc o_def*)
also have . . . = (∑ *t*←*?us*. *7*∗*size t*) + *length ts*
by(*simp add*: *sum_tree_list_children*)
also have . . . ≤ (∑ *t*←*ts*. *7*∗*size t*) + *length ts*
by(*simp add*: *sum_list_filter_le_nat*)
also have . . . = (∑ *t*←*ts*. *7* ∗ *size t* + *1*)
by(*simp add*: *sum_list_Suc*)
finally show *?case* **.**
qed
qed

end

40 Tries via Functions

theory *Trie_Fun*
imports
Set_Specs
begin

A trie where each node maps a key to sub-tries via a function. Nice abstract model. Not efficient because of the function space.

datatype ′*a trie* = *Nd bool* ′*a* ⇒ ′*a trie option*

definition *empty* :: ′*a trie* **where**
[*simp*]: *empty* = *Nd False* (λ_. *None*)

fun *isin* :: *'a trie* ⇒ *'a list* ⇒ *bool* **where**
isin (*Nd b m*) [] = *b* |
isin (*Nd b m*) (*k* # *xs*) = (*case m k of None* ⇒ *False* | *Some t* ⇒ *isin t xs*)

fun *insert* :: *'a list* ⇒ *'a trie* ⇒ *'a trie* **where**
insert [] (*Nd b m*) = *Nd True m* |
insert (*x#xs*) (*Nd b m*) =
  (*let s* = (*case m x of None* ⇒ *empty* | *Some t* ⇒ *t*) *in Nd b* (*m*(*x* := *Some*(*insert xs s*))))

fun *delete* :: *'a list* ⇒ *'a trie* ⇒ *'a trie* **where**
delete [] (*Nd b m*) = *Nd False m* |
delete (*x#xs*) (*Nd b m*) = *Nd b*
  (*case m x of*
    None ⇒ *m* |
    Some t ⇒ *m*(*x* := *Some*(*delete xs t*)))

Use (a tuned version of) *isin* as an abstraction function:

lemma *isin_case*: *isin* (*Nd b m*) *xs* =
  (*case xs of*
  [] ⇒ *b* |
  x # *ys* ⇒ (*case m x of None* ⇒ *False* | *Some t* ⇒ *isin t ys*))
by(*cases xs*)*auto*

definition *set* :: *'a trie* ⇒ *'a list set* **where**
[*simp*]: *set t* = {*xs*. *isin t xs*}

lemma *isin_set*: *isin t xs* = (*xs* ∈ *set t*)
by *simp*

lemma *set_insert*: *set* (*insert xs t*) = *set t* ∪ {*xs*}
by (*induction xs t rule*: *insert.induct*)
  (*auto simp*: *isin_case split*!: *if_splits option.splits list.splits*)

lemma *set_delete*: *set* (*delete xs t*) = *set t* − {*xs*}
by (*induction xs t rule*: *delete.induct*)
  (*auto simp*: *isin_case split*!: *if_splits option.splits list.splits*)

interpretation *S*: *Set*
where *empty* = *empty* **and** *isin* = *isin* **and** *insert* = *insert* **and** *delete* = *delete*
and *set* = *set* **and** *invar* = λ_. *True*
proof (*standard*, *goal_cases*)

case *1* **show** *?case* **by** (*simp add*: *isin_case split*: *list.split*)
next
case *2* **show** *?case* **by**(*rule isin_set*)
next
case *3* **show** *?case* **by**(*rule set_insert*)
next
case *4* **show** *?case* **by**(*rule set_delete*)
qed (*rule TrueI*)+

end

41 Tries via Search Trees

theory *Trie_Map*
imports
Tree_Map
Trie_Fun
begin

An implementation of tries for an arbitrary alphabet *'a* where the mapping from an element of type *'a* to the sub-trie is implemented by a binary search tree. Although this implementation uses maps implemented by red-black trees it works for any implementation of maps.

This is an implementation of the "ternary search trees" by Bentley and Sedgewick [SODA 1997, Dr. Dobbs 1998]. The name derives from the fact that a node in the BST can now be drawn to have 3 children, where the middle child is the sub-trie that the node maps its key to. Hence the name *trie3*.

Example from https://en.wikipedia.org/wiki/Ternary_search_tree#Description:

c / | a u h | | | t. t e. u / / | / | s. p. e. i. s.

Characters with a dot are final. Thus the tree represents the set of strings "cute","cup","at","as","he","us" and "i".

datatype *'a trie3* = *Nd3 bool* (*'a* ∗ *'a trie3*) *tree*

In principle one should be able to given an implementation of tries once and for all for any map implementation and not just for a specific one (unbalanced trees) as done here. But because the map (*tree*) is used in a datatype, the HOL type system does not support this.

However, the development below works verbatim for any map implementation, eg *RBT_Map*, and not just *Tree_Map*, except for the termination lemma *lookup_size*.

term *size_tree*
lemma *lookup_size*[*termination_simp*]:
fixes *t* :: (*'a*::*linorder* ∗ *'a trie3*) *tree*

shows *lookup t a = Some b ⟹ size b < Suc (size_tree (λab. Suc (size (snd(ab)))) t)*
apply(*induction t a rule*: *lookup.induct*)
apply(*auto split*: *if_splits*)
done

definition *empty3* :: *'a trie3* **where**
[*simp*]: *empty3 = Nd3 False Leaf*

fun *isin3* :: (*'a::linorder*) *trie3 ⇒ 'a list ⇒ bool* **where**
isin3 (Nd3 b m) [] = b |
isin3 (Nd3 b m) (x # xs) = (case lookup m x of None ⇒ False | Some t ⇒ isin3 t xs)

fun *insert3* :: (*'a::linorder*) *list ⇒ 'a trie3 ⇒ 'a trie3* **where**
insert3 [] (Nd3 b m) = Nd3 True m |
insert3 (x#xs) (Nd3 b m) =
  Nd3 b (update x (insert3 xs (case lookup m x of None ⇒ empty3 | Some t ⇒ t)) m)

fun *delete3* :: (*'a::linorder*) *list ⇒ 'a trie3 ⇒ 'a trie3* **where**
delete3 [] (Nd3 b m) = Nd3 False m |
delete3 (x#xs) (Nd3 b m) = Nd3 b
   (case lookup m x of
      None ⇒ m |
      Some t ⇒ update x (delete3 xs t) m)

41.1 Correctness

Proof by stepwise refinement. First abs3tract to type *'a trie*.

fun *abs3* :: *'a::linorder trie3 ⇒ 'a trie* **where**
abs3 (Nd3 b t) = Nd b (λa. map_option abs3 (lookup t a))

fun *invar3* :: (*'a::linorder*)*trie3 ⇒ bool* **where**
invar3 (Nd3 b m) = (M.invar m ∧ (∀ a t. lookup m a = Some t ⟶ invar3 t))

lemma *isin_abs3*: *isin3 t xs = isin (abs3 t) xs*
apply(*induction t xs rule*: *isin3.induct*)
apply(*auto split*: *option.split*)
done

lemma *abs3_insert3*: *invar3 t ⟹ abs3(insert3 xs t) = insert xs (abs3 t)*

apply(*induction xs t rule*: *insert3.induct*)
apply(*auto simp*: *M.map_specs Tree_Set.empty_def*[*symmetric*] *split*: *option.split*)
done

lemma *abs3_delete3*: *invar3 t* ⟹ *abs3*(*delete3 xs t*) = *delete xs* (*abs3 t*)
apply(*induction xs t rule*: *delete3.induct*)
apply(*auto simp*: *M.map_specs split*: *option.split*)
done

lemma *invar3_insert3*: *invar3 t* ⟹ *invar3* (*insert3 xs t*)
apply(*induction xs t rule*: *insert3.induct*)
apply(*auto simp*: *M.map_specs Tree_Set.empty_def*[*symmetric*] *split*: *option.split*)
done

lemma *invar3_delete3*: *invar3 t* ⟹ *invar3* (*delete3 xs t*)
apply(*induction xs t rule*: *delete3.induct*)
apply(*auto simp*: *M.map_specs split*: *option.split*)
done

Overall correctness w.r.t. the *Set* ADT:

interpretation *S2*: *Set*
where *empty* = *empty3* **and** *isin* = *isin3* **and** *insert* = *insert3* **and** *delete* = *delete3*
and *set* = *set o abs3* **and** *invar* = *invar3*
proof (*standard*, *goal_cases*)
  case *1* **show** *?case* **by** (*simp add*: *isin_case split*: *list.split*)
next
  case *2* **thus** *?case* **by** (*simp add*: *isin_abs3*)
next
  case *3* **thus** *?case* **by** (*simp add*: *set_insert abs3_insert3 del*: *set_def*)
next
  case *4* **thus** *?case* **by** (*simp add*: *set_delete abs3_delete3 del*: *set_def*)
next
  case *5* **thus** *?case* **by** (*simp add*: *M.map_specs Tree_Set.empty_def*[*symmetric*])
next
  case *6* **thus** *?case* **by** (*simp add*: *invar3_insert3*)
next
  case *7* **thus** *?case* **by** (*simp add*: *invar3_delete3*)
qed

end

42 Binary Tries and Patricia Tries

theory *Tries_Binary*
imports *Set_Specs*
begin

hide_const (**open**) *insert*

declare *Let_def*[*simp*]

fun *sel2* :: *bool* ⇒ *'a* ∗ *'a* ⇒ *'a* **where**
sel2 b (*a1*,*a2*) = (*if b then a2 else a1*)

fun *mod2* :: (*'a* ⇒ *'a*) ⇒ *bool* ⇒ *'a* ∗ *'a* ⇒ *'a* ∗ *'a* **where**
mod2 f b (*a1*,*a2*) = (*if b then* (*a1*,*f a2*) *else* (*f a1*,*a2*))

42.1 Trie

datatype *trie* = *Lf* | *Nd bool trie* ∗ *trie*

definition *empty* :: *trie* **where**
[*simp*]: *empty* = *Lf*

fun *isin* :: *trie* ⇒ *bool list* ⇒ *bool* **where**
isin Lf ks = *False* |
isin (*Nd b lr*) *ks* =
  (*case ks of*
    [] ⇒ *b* |
    k#ks ⇒ *isin* (*sel2 k lr*) *ks*)

fun *insert* :: *bool list* ⇒ *trie* ⇒ *trie* **where**
insert [] *Lf* = *Nd True* (*Lf*,*Lf*) |
insert [] (*Nd b lr*) = *Nd True lr* |
insert (*k#ks*) *Lf* = *Nd False* (*mod2* (*insert ks*) *k* (*Lf*,*Lf*)) |
insert (*k#ks*) (*Nd b lr*) = *Nd b* (*mod2* (*insert ks*) *k lr*)

lemma *isin_insert*: *isin* (*insert xs t*) *ys* = (*xs* = *ys* ∨ *isin t ys*)
apply(*induction xs t arbitrary*: *ys rule*: *insert.induct*)
apply (*auto split*: *list.splits if_splits*)
done

A simple implementation of delete; does not shrink the trie!

fun *delete0* :: *bool list* ⇒ *trie* ⇒ *trie* **where**
delete0 ks Lf = *Lf* |

delete0 ks (*Nd b lr*) =
  (*case ks of*
    [] ⇒ *Nd False lr* |
    k#*ks*′ ⇒ *Nd b* (*mod2* (*delete0 ks*′) *k lr*))

lemma *isin_delete0*: *isin* (*delete0 as t*) *bs* = (*as* ≠ *bs* ∧ *isin t bs*)
apply(*induction as t arbitrary*: *bs rule*: *delete0.induct*)
apply (*auto split*: *list.splits if_splits*)
done

Now deletion with shrinking:

fun *node* :: *bool* ⇒ *trie* ∗ *trie* ⇒ *trie* **where**
node b lr = (*if* ¬ *b* ∧ *lr* = (*Lf*,*Lf*) *then Lf else Nd b lr*)

fun *delete* :: *bool list* ⇒ *trie* ⇒ *trie* **where**
delete ks Lf = *Lf* |
delete ks (*Nd b lr*) =
  (*case ks of*
    [] ⇒ *node False lr* |
    k#*ks*′ ⇒ *node b* (*mod2* (*delete ks*′) *k lr*))

lemma *isin_delete*: *isin* (*delete xs t*) *ys* = (*xs* ≠ *ys* ∧ *isin t ys*)
apply(*induction xs t arbitrary*: *ys rule*: *delete.induct*)
 apply *simp*
apply (*auto split*: *list.splits if_splits*)
  apply (*metis isin.simps*(*1*))
 apply (*metis isin.simps*(*1*))
  done

definition *set_trie* :: *trie* ⇒ *bool list set* **where**
set_trie t = {*xs*. *isin t xs*}

lemma *set_trie_empty*: *set_trie empty* = {}
by(*simp add*: *set_trie_def*)

lemma *set_trie_isin*: *isin t xs* = (*xs* ∈ *set_trie t*)
by(*simp add*: *set_trie_def*)

lemma *set_trie_insert*: *set_trie*(*insert xs t*) = *set_trie t* ∪ {*xs*}
by(*auto simp add*: *isin_insert set_trie_def*)

lemma *set_trie_delete*: *set_trie*(*delete xs t*) = *set_trie t* − {*xs*}
by(*auto simp add*: *isin_delete set_trie_def*)

Invariant: tries are fully shrunk:

```
fun invar where
invar Lf = True |
invar (Nd b (l,r)) = (invar l ∧ invar r ∧ (l = Lf ∧ r = Lf ⟶ b))

lemma insert_Lf: insert xs t ≠ Lf
using insert.elims by blast

lemma invar_insert: invar t ⟹ invar(insert xs t)
proof(induction xs t rule: insert.induct)
  case 1 thus ?case by simp
next
  case (2 b lr)
  thus ?case by(cases lr; simp)
next
  case (3 k ks)
  thus ?case by(simp; cases ks; auto)
next
  case (4 k ks b lr)
  then show ?case by(cases lr; auto simp: insert_Lf)
qed

lemma invar_delete: invar t ⟹ invar(delete xs t)
proof(induction t arbitrary: xs)
  case Lf thus ?case by simp
next
  case (Nd b lr)
  thus ?case by(cases lr)(auto split: list.split)
qed

interpretation S: Set
where empty = empty and isin = isin and insert = insert and delete =
delete
and set = set_trie and invar = invar
proof (standard, goal_cases)
  case 1 show ?case by (rule set_trie_empty)
next
  case 2 show ?case by(rule set_trie_isin)
next
  case 3 thus ?case by(auto simp: set_trie_insert)
next
  case 4 show ?case by(rule set_trie_delete)
next
  case 5 show ?case by(simp)
next
```

case *6* **thus** *?case* **by**(*rule invar_insert*)
next
case *7* **thus** *?case* **by**(*rule invar_delete*)
qed

42.2 Patricia Trie

datatype *trieP* = *LfP* | *NdP bool list bool trieP* ∗ *trieP*

Fully shrunk:

fun *invarP* **where**
invarP LfP = *True* |
$invarP\ (NdP\ ps\ b\ (l,r)) = (invarP\ l \wedge invarP\ r \wedge (l = LfP \vee r = LfP \longrightarrow b))$

fun *isinP* :: $trieP \Rightarrow bool\ list \Rightarrow bool$ **where**
isinP LfP ks = *False* |
isinP (*NdP ps b lr*) *ks* =
(*let n* = *length ps in*
if ps = *take n ks*
then case drop n ks of [] ⇒ *b* | $k\#ks' \Rightarrow isinP\ (sel2\ k\ lr)\ ks'$
else False)

definition *emptyP* :: *trieP* **where**
[*simp*]: *emptyP* = *LfP*

fun *lcp* :: $'a\ list \Rightarrow 'a\ list \Rightarrow 'a\ list \times 'a\ list \times 'a\ list$ **where**
lcp [] *ys* = ([],[],*ys*) |
lcp xs [] = ([],*xs*,[]) |
lcp (*x#xs*) (*y#ys*) =
($if\ x \neq y\ then\ ([],x\#xs,y\#ys)$
$else\ let\ (ps,xs',ys') = lcp\ xs\ ys\ in\ (x\#ps,xs',ys'))$

lemma *mod2_cong*[*fundef_cong*]:
$[\![lr = lr'; k = k'; \bigwedge a\ b.\ lr'=(a,b) \Longrightarrow f\ (a) = f'\ (a)\ ; \bigwedge a\ b.\ lr'=(a,b) \Longrightarrow f\ (b) = f'\ (b)\ ]\!]$
$\Longrightarrow mod2\ f\ k\ lr = mod2\ f'\ k'\ lr'$
by(*cases lr*, *cases lr'*, *auto*)

fun *insertP* :: $bool\ list \Rightarrow trieP \Rightarrow trieP$ **where**
insertP ks LfP = *NdP ks True* (*LfP*,*LfP*) |
insertP ks (*NdP ps b lr*) =

(*case lcp ks ps of*
  (*qs*, *k*#*ks*′, *p*#*ps*′) ⇒
   let tp = *NdP ps*′ *b lr*; *tk* = *NdP ks*′ *True* (*LfP*,*LfP*) *in*
   NdP qs False (*if k then* (*tp*,*tk*) *else* (*tk*,*tp*)) |
  (*qs*, *k*#*ks*′, []) ⇒
   NdP ps b (*mod2* (*insertP ks*′) *k lr*) |
  (*qs*, [], *p*#*ps*′) ⇒
   let t = *NdP ps*′ *b lr in*
   NdP qs True (*if p then* (*LfP*,*t*) *else* (*t*,*LfP*)) |
  (*qs*,[],[]) ⇒ *NdP ps True lr*)

Smart constructor that shrinks:

definition *nodeP* :: *bool list* ⇒ *bool* ⇒ *trieP* ∗ *trieP* ⇒ *trieP* **where**
nodeP ps b lr =
(*if b then NdP ps b lr*
 else case lr of
  (*LfP*,*LfP*) ⇒ *LfP* |
  (*LfP*, *NdP ks b lr*) ⇒ *NdP* (*ps* @ *True* # *ks*) *b lr* |
  (*NdP ks b lr*, *LfP*) ⇒ *NdP* (*ps* @ *False* # *ks*) *b lr* |
  _ ⇒ *NdP ps b lr*)

fun *deleteP* :: *bool list* ⇒ *trieP* ⇒ *trieP* **where**
deleteP ks LfP = *LfP* |
deleteP ks (*NdP ps b lr*) =
 (*case lcp ks ps of*
   (_, _, _#_) ⇒ *NdP ps b lr* |
   (_, *k*#*ks*′, []) ⇒ *nodeP ps b* (*mod2* (*deleteP ks*′) *k lr*) |
   (_, [], []) ⇒ *nodeP ps False lr*)

42.2.1 Functional Correctness

First step: *trieP* implements *trie* via the abstraction function *abs_trieP*:

fun *prefix_trie* :: *bool list* ⇒ *trie* ⇒ *trie* **where**
prefix_trie [] *t* = *t* |
prefix_trie (*k*#*ks*) *t* =
 (*let t*′ = *prefix_trie ks t in Nd False* (*if k then* (*Lf*,*t*′) *else* (*t*′,*Lf*)))

fun *abs_trieP* :: *trieP* ⇒ *trie* **where**
abs_trieP LfP = *Lf* |
abs_trieP (*NdP ps b* (*l*,*r*)) = *prefix_trie ps* (*Nd b* (*abs_trieP l*, *abs_trieP r*))

Correctness of *isinP*:

lemma *isin_prefix_trie*:

isin (*prefix_trie ps t*) *ks*
= (*ps* = *take* (*length ps*) *ks* ∧ *isin t* (*drop* (*length ps*) *ks*))
apply(*induction ps arbitrary*: *ks*)
apply(*auto split*: *list.split*)
done

lemma *abs_trieP_isinP*:
isinP t ks = *isin* (*abs_trieP t*) *ks*
apply(*induction t arbitrary*: *ks rule*: *abs_trieP.induct*)
apply(*auto simp*: *isin_prefix_trie split*: *list.split*)
done

Correctness of *insertP*:

lemma *prefix_trie_Lfs*: *prefix_trie ks* (*Nd True* (*Lf*,*Lf*)) = *insert ks Lf*
apply(*induction ks*)
apply *auto*
done

lemma *insert_prefix_trie_same*:
insert ps (*prefix_trie ps* (*Nd b lr*)) = *prefix_trie ps* (*Nd True lr*)
apply(*induction ps*)
apply *auto*
done

lemma *insert_append*: *insert* (*ks* @ *ks'*) (*prefix_trie ks t*) = *prefix_trie ks* (*insert ks' t*)
apply(*induction ks*)
apply *auto*
done

lemma *prefix_trie_append*: *prefix_trie* (*ps* @ *qs*) *t* = *prefix_trie ps* (*prefix_trie qs t*)
apply(*induction ps*)
apply *auto*
done

lemma *lcp_if*: *lcp ks ps* = (*qs*, *ks'*, *ps'*) ⟹
ks = *qs* @ *ks'* ∧ *ps* = *qs* @ *ps'* ∧ (*ks'* ≠ [] ∧ *ps'* ≠ [] ⟶ *hd ks'* ≠ *hd ps'*)
apply(*induction ks ps arbitrary*: *qs ks' ps' rule*: *lcp.induct*)
apply(*auto split*: *prod.splits if_splits*)
done

lemma *abs_trieP_insertP*:
abs_trieP (*insertP ks t*) = *insert ks* (*abs_trieP t*)

apply(*induction t arbitrary*: *ks*)
apply(*auto simp*: *prefix_trie_Lfs insert_prefix_trie_same insert_append prefix_trie_append*
dest!: *lcp_if split*: *list.split prod.split if_splits*)
done

Correctness of *deleteP*:

lemma *prefix_trie_Lf*: *prefix_trie xs t* = *Lf* ⟷ *xs* = [] ∧ *t* = *Lf*
by(*cases xs*)(*auto*)

lemma *abs_trieP_Lf*: *abs_trieP t* = *Lf* ⟷ *t* = *LfP*
by(*cases t*) (*auto simp*: *prefix_trie_Lf*)

lemma *delete_prefix_trie*:
delete xs (*prefix_trie xs* (*Nd b* (*l*,*r*)))
= (*if* (*l*,*r*) = (*Lf*,*Lf*) *then Lf else prefix_trie xs* (*Nd False* (*l*,*r*)))
by(*induction xs*)(*auto simp*: *prefix_trie_Lf*)

lemma *delete_append_prefix_trie*:
delete (*xs* @ *ys*) (*prefix_trie xs t*)
= (*if delete ys t* = *Lf then Lf else prefix_trie xs* (*delete ys t*))
by(*induction xs*)(*auto simp*: *prefix_trie_Lf*)

lemma *nodeP_LfP2*: *nodeP xs False* (*LfP*, *LfP*) = *LfP*
by(*simp add*: *nodeP_def*)

Some non-inductive aux. lemmas:

lemma *abs_trieP_nodeP*: *a*≠*LfP* ∨ *b* ≠ *LfP* ⟹
abs_trieP (*nodeP xs f* (*a*, *b*)) = *prefix_trie xs* (*Nd f* (*abs_trieP a*, *abs_trieP b*))
by(*auto simp add*: *nodeP_def prefix_trie_append split*: *trieP.split*)

lemma *nodeP_True*: *nodeP ps True lr* = *NdP ps True lr*
by(*simp add*: *nodeP_def*)

lemma *delete_abs_trieP*:
delete ks (*abs_trieP t*) = *abs_trieP* (*deleteP ks t*)
apply(*induction t arbitrary*: *ks*)
apply(*auto simp*: *delete_prefix_trie delete_append_prefix_trie*
prefix_trie_append prefix_trie_Lf abs_trieP_Lf nodeP_LfP2 abs_trieP_nodeP nodeP_True
dest!: *lcp_if split*: *if_splits list.split prod.split*)
done

Invariant preservation:

lemma *insertP_LfP*: *insertP xs t* ≠ *LfP*
by(*cases t*)(*auto split*: *prod.split list.split*)

lemma *invarP_insertP*: *invarP t* ⟹ *invarP*(*insertP xs t*)
proof(*induction t arbitrary*: *xs*)
  case *LfP* **thus** *?case* **by** *simp*
next
  case (*NdP bs b lr*)
  then show *?case*
    by(*cases lr*)(*auto simp*: *insertP_LfP split*: *prod.split list.split*)
qed

lemma *invarP_nodeP*: ⟦ *invarP t1*; *invarP t2*⟧ ⟹ *invarP* (*nodeP xs b* (*t1*, *t2*))
by (*auto simp add*: *nodeP_def split*: *trieP.split*)

lemma *invarP_deleteP*: *invarP t* ⟹ *invarP*(*deleteP xs t*)
proof(*induction t arbitrary*: *xs*)
  case *LfP* **thus** *?case* **by** *simp*
next
  case (*NdP ks b lr*)
  thus *?case* **by**(*cases lr*)(*auto simp*: *invarP_nodeP split*: *prod.split list.split*)
qed

The overall correctness proof. Simply composes correctness lemmas.

definition *set_trieP* :: *trieP* ⇒ *bool list set* **where**
set_trieP = *set_trie o abs_trieP*

lemma *isinP_set_trieP*: *isinP t xs* = (*xs* ∈ *set_trieP t*)
by(*simp add*: *abs_trieP_isinP set_trie_isin set_trieP_def*)

lemma *set_trieP_insertP*: *set_trieP* (*insertP xs t*) = *set_trieP t* ∪ {*xs*}
by(*simp add*: *abs_trieP_insertP set_trie_insert set_trieP_def*)

lemma *set_trieP_deleteP*: *set_trieP* (*deleteP xs t*) = *set_trieP t* − {*xs*}
by(*auto simp*: *set_trie_delete set_trieP_def simp flip*: *delete_abs_trieP*)

interpretation *SP*: *Set*
where *empty* = *emptyP* **and** *isin* = *isinP* **and** *insert* = *insertP* **and** *delete* = *deleteP*
and *set* = *set_trieP* **and** *invar* = *invarP*
proof (*standard*, *goal_cases*)
  case *1* **show** *?case* **by** (*simp add*: *set_trieP_def set_trie_def*)

next
  case *2* **show** *?case* **by**(*rule isinP_set_trieP*)
next
  case *3* **thus** *?case* **by** (*auto simp*: *set_trieP_insertP*)
next
  case *4* **thus** *?case* **by**(*auto simp*: *set_trieP_deleteP*)
next
  case *5* **thus** *?case* **by**(*simp*)
next
  case *6* **thus** *?case* **by**(*rule invarP_insertP*)
next
  case *7* **thus** *?case* **by**(*rule invarP_deleteP*)
qed

end

43 Queue Specification

theory *Queue_Spec*
imports *Main*
begin

The basic queue interface with *list*-based specification:

locale *Queue* =
fixes *empty* :: *'q*
fixes *enq* :: *'a* ⇒ *'q* ⇒ *'q*
fixes *first* :: *'q* ⇒ *'a*
fixes *deq* :: *'q* ⇒ *'q*
fixes *is_empty* :: *'q* ⇒ *bool*
fixes *list* :: *'q* ⇒ *'a list*
fixes *invar* :: *'q* ⇒ *bool*
assumes *list_empty*: *list empty* = []
assumes *list_enq*: *invar q* ⟹ *list*(*enq x q*) = *list q* @ [*x*]
assumes *list_deq*: *invar q* ⟹ *list*(*deq q*) = *tl*(*list q*)
assumes *list_first*: *invar q* ⟹ ¬ *list q* = [] ⟹ *first q* = *hd*(*list q*)
assumes *list_is_empty*: *invar q* ⟹ *is_empty q* = (*list q* = [])
assumes *invar_empty*: *invar empty*
assumes *invar_enq*: *invar q* ⟹ *invar*(*enq x q*)
assumes *invar_deq*: *invar q* ⟹ *invar*(*deq q*)

end
theory *Reverse*
imports *Main*
begin

fun *T_append* :: *'a list* ⇒ *'a list* ⇒ *nat* **where**
T_append [] *ys* = *1* |
T_append (*x*#*xs*) *ys* = *T_append xs ys* + *1*

fun *T_rev* :: *'a list* ⇒ *nat* **where**
T_rev [] = *1* |
T_rev (*x*#*xs*) = *T_rev xs* + *T_append* (*rev xs*) [*x*] + *1*

lemma *T_append*: *T_append xs ys* = *length xs* + *1*
by(*induction xs*) *auto*

lemma *T_rev*: *T_rev xs* ≤ (*length xs* + *1*)^*2*
by(*induction xs*) (*auto simp*: *T_append power2_eq_square*)

fun *itrev* :: *'a list* ⇒ *'a list* ⇒ *'a list* **where**
itrev [] *ys* = *ys* |
itrev (*x*#*xs*) *ys* = *itrev xs* (*x* # *ys*)

lemma *itrev*: *itrev xs ys* = *rev xs* @ *ys*
by(*induction xs arbitrary*: *ys*) *auto*

lemma *itrev_Nil*: *itrev xs* [] = *rev xs*
by(*simp add*: *itrev*)

fun *T_itrev* :: *'a list* ⇒ *'a list* ⇒ *nat* **where**
T_itrev [] *ys* = *1* |
T_itrev (*x*#*xs*) *ys* = *T_itrev xs* (*x* # *ys*) + *1*

lemma *T_itrev*: *T_itrev xs ys* = *length xs* + *1*
by(*induction xs arbitrary*: *ys*) *auto*

end

44 Queue Implementation via 2 Lists

theory *Queue_2Lists*
imports
  Queue_Spec
  Reverse
begin

Definitions:

type_synonym *'a queue* = *'a list* × *'a list*

fun *norm* :: *'a queue* ⇒ *'a queue* **where**
norm (*fs*,*rs*) = (*if fs* = [] *then* (*itrev rs* [], []) *else* (*fs*,*rs*))

fun *enq* :: *'a* ⇒ *'a queue* ⇒ *'a queue* **where**
enq a (*fs*,*rs*) = *norm*(*fs*, *a* # *rs*)

fun *deq* :: *'a queue* ⇒ *'a queue* **where**
deq (*fs*,*rs*) = (*if fs* = [] *then* (*fs*,*rs*) *else norm*(*tl fs*,*rs*))

fun *first* :: *'a queue* ⇒ *'a* **where**
first (*a* # *fs*,*rs*) = *a*

fun *is_empty* :: *'a queue* ⇒ *bool* **where**
is_empty (*fs*,*rs*) = (*fs* = [])

fun *list* :: *'a queue* ⇒ *'a list* **where**
list (*fs*,*rs*) = *fs* @ *rev rs*

fun *invar* :: *'a queue* ⇒ *bool* **where**
invar (*fs*,*rs*) = (*fs* = [] ⟶ *rs* = [])

Implementation correctness:

interpretation *Queue*
where *empty* = ([],[]) **and** *enq* = *enq* **and** *deq* = *deq* **and** *first* = *first*
and *is_empty* = *is_empty* **and** *list* = *list* **and** *invar* = *invar*
proof (*standard*, *goal_cases*)
  case *1* **show** *?case* **by** (*simp*)
next
  case (*2 q*) **thus** *?case* **by**(*cases q*) (*simp*)
next
  case (*3 q*) **thus** *?case* **by**(*cases q*) (*simp add*: *itrev_Nil*)
next
  case (*4 q*) **thus** *?case* **by**(*cases q*) (*auto simp*: *neq_Nil_conv*)
next
  case (*5 q*) **thus** *?case* **by**(*cases q*) (*auto*)
next
  case *6* **show** *?case* **by**(*simp*)
next
  case (*7 q*) **thus** *?case* **by**(*cases q*) (*simp*)
next
  case (*8 q*) **thus** *?case* **by**(*cases q*) (*simp*)
qed

Running times:

fun *T_norm* :: *'a queue* ⇒ *nat* **where**
T_norm (*fs*,*rs*) = (*if fs* = [] *then T_itrev rs* [] *else 0*) + *1*

fun *T_enq* :: *'a* ⇒ *'a queue* ⇒ *nat* **where**
T_enq a (*fs*,*rs*) = *T_norm*(*fs*, *a* # *rs*) + *1*

fun *T_deq* :: *'a queue* ⇒ *nat* **where**
T_deq (*fs*,*rs*) = (*if fs* = [] *then 0 else T_norm*(*tl fs*,*rs*)) + *1*

fun *T_first* :: *'a queue* ⇒ *nat* **where**
T_first (*a* # *fs*,*rs*) = *1*

fun *T_is_empty* :: *'a queue* ⇒ *nat* **where**
T_is_empty (*fs*,*rs*) = *1*

Amortized running times:

fun Φ :: *'a queue* ⇒ *nat* **where**
Φ(*fs*,*rs*) = *length rs*

lemma *a_enq*: *T_enq a* (*fs*,*rs*) + Φ(*enq a* (*fs*,*rs*)) − Φ(*fs*,*rs*) ≤ *4*
by(*auto simp*: *T_itrev*)

lemma *a_deq*: *T_deq* (*fs*,*rs*) + Φ(*deq* (*fs*,*rs*)) − Φ(*fs*,*rs*) ≤ *3*
by(*auto simp*: *T_itrev*)

end

45 Priority Queue Specifications

theory *Priority_Queue_Specs*
imports *HOL−Library.Multiset*
begin

Priority queue interface + specification:

locale *Priority_Queue* =
fixes *empty* :: *'q*
and *is_empty* :: *'q* ⇒ *bool*
and *insert* :: *'a*::*linorder* ⇒ *'q* ⇒ *'q*
and *get_min* :: *'q* ⇒ *'a*
and *del_min* :: *'q* ⇒ *'q*
and *invar* :: *'q* ⇒ *bool*
and *mset* :: *'q* ⇒ *'a multiset*
assumes *mset_empty*: *mset empty* = {#}
and *is_empty*: *invar q* ⟹ *is_empty q* = (*mset q* = {#})

and *mset_insert*: *invar q* ⟹ *mset* (*insert x q*) = *mset q* + {#*x*#}
and *mset_del_min*: *invar q* ⟹ *mset q* ≠ {#} ⟹
mset (*del_min q*) = *mset q* − {# *get_min q* #}
and *mset_get_min*: *invar q* ⟹ *mset q* ≠ {#} ⟹ *get_min q* = *Min_mset* (*mset q*)
and *invar_empty*: *invar empty*
and *invar_insert*: *invar q* ⟹ *invar* (*insert x q*)
and *invar_del_min*: *invar q* ⟹ *mset q* ≠ {#} ⟹ *invar* (*del_min q*)

Extend locale with *merge*. Need to enforce that ′*q* is the same in both locales.

locale *Priority_Queue_Merge* = *Priority_Queue* **where** *empty* = *empty* **for** *empty* :: ′*q* +
fixes *merge* :: ′*q* ⇒ ′*q* ⇒ ′*q*
assumes *mset_merge*: ⟦ *invar q1*; *invar q2* ⟧ ⟹ *mset* (*merge q1 q2*) = *mset q1* + *mset q2*
and *invar_merge*: ⟦ *invar q1*; *invar q2* ⟧ ⟹ *invar* (*merge q1 q2*)

end

46 Heaps

theory *Heaps*
imports
HOL−Library.Tree_Multiset
Priority_Queue_Specs
begin

Heap = priority queue on trees:

locale *Heap* =
fixes *insert* :: (′*a*::*linorder*) ⇒ ′*a tree* ⇒ ′*a tree*
and *del_min* :: ′*a tree* ⇒ ′*a tree*
assumes *mset_insert*: *heap q* ⟹ *mset_tree* (*insert x q*) = {#*x*#} + *mset_tree q*
and *mset_del_min*: ⟦ *heap q*; *q* ≠ *Leaf* ⟧ ⟹ *mset_tree* (*del_min q*) = *mset_tree q* − {#*value q*#}
and *heap_insert*: *heap q* ⟹ *heap*(*insert x q*)
and *heap_del_min*: *heap q* ⟹ *heap*(*del_min q*)
begin

definition *empty* :: ′*a tree* **where**
empty = *Leaf*

fun *is_empty* :: ′*a tree* ⇒ *bool* **where**

is_empty t = (*t* = *Leaf*)

sublocale *Priority_Queue* **where** *empty* = *empty* **and** *is_empty* = *is_empty*
and *insert* = *insert*
and *get_min* = *value* **and** *del_min* = *del_min* **and** *invar* = *heap* **and** *mset* = *mset_tree*
proof (*standard*, *goal_cases*)
  case *1* **thus** *?case* **by** (*simp add*: *empty_def*)
next
  case *2* **thus** *?case* **by**(*auto*)
next
  case *3* **thus** *?case* **by**(*simp add*: *mset_insert*)
next
  case *4* **thus** *?case* **by**(*simp add*: *mset_del_min*)
next
  case *5* **thus** *?case* **by**(*auto simp*: *neq_Leaf_iff Min_insert2 simp del*: *Un_iff*)
next
  case *6* **thus** *?case* **by**(*simp add*: *empty_def*)
next
  case *7* **thus** *?case* **by**(*simp add*: *heap_insert*)
next
  case *8* **thus** *?case* **by**(*simp add*: *heap_del_min*)
qed

end

Once you have *merge*, *insert* and *del_min* are easy:

locale *Heap_Merge* =
fixes *merge* :: ′*a*::*linorder tree* ⇒ ′*a tree* ⇒ ′*a tree*
assumes *mset_merge*: ⟦ *heap q1*; *heap q2* ⟧ ⟹ *mset_tree* (*merge q1 q2*) = *mset_tree q1* + *mset_tree q2*
and *invar_merge*: ⟦ *heap q1*; *heap q2* ⟧ ⟹ *heap* (*merge q1 q2*)
begin

fun *insert* :: ′*a* ⇒ ′*a tree* ⇒ ′*a tree* **where**
insert x t = *merge* (*Node Leaf x Leaf*) *t*

fun *del_min* :: ′*a tree* ⇒ ′*a tree* **where**
del_min Leaf = *Leaf* |
del_min (*Node l x r*) = *merge l r*

interpretation *Heap insert del_min*
proof(*standard*, *goal_cases*)

case *1* **thus** *?case* **by**(*simp add*:*mset_merge*)
next
case (*2 q*) **thus** *?case* **by**(*cases q*)(*auto simp*: *mset_merge*)
next
case *3* **thus** *?case* **by** (*simp add*: *invar_merge*)
next
case (*4 q*) **thus** *?case* **by** (*cases q*)(*auto simp*: *invar_merge*)
qed

sublocale *PQM*: *Priority_Queue_Merge* **where** *empty* = *empty* **and** *is_empty* = *is_empty* **and** *insert* = *insert*
and *get_min* = *value* **and** *del_min* = *del_min* **and** *invar* = *heap* **and** *mset* = *mset_tree* **and** *merge* = *merge*
proof(*standard*, *goal_cases*)
case *1* **thus** *?case* **by** (*simp add*: *mset_merge*)
next
case *2* **thus** *?case* **by** (*simp add*: *invar_merge*)
qed

end

end

47 Leftist Heap

theory *Leftist_Heap*
imports
HOL−Library.Pattern_Aliases
Tree2
Priority_Queue_Specs
Complex_Main
begin

fun *mset_tree* :: (′*a*∗′*b*) *tree* ⇒ ′*a multiset* **where**
mset_tree Leaf = {#} |
mset_tree (*Node l* (*a*, _) *r*) = {#*a*#} + *mset_tree l* + *mset_tree r*

type_synonym ′*a lheap* = (′*a*∗*nat*)*tree*

fun *mht* :: ′*a lheap* ⇒ *nat* **where**
mht Leaf = *0* |
mht (*Node* _ (_, *n*) _) = *n*

The invariants:

fun (**in** *linorder*) *heap* :: (*'a***'b*) *tree* ⇒ *bool* **where**
heap Leaf = *True* |
heap (*Node l* (*m*, _) *r*) =
  ((∀ *x* ∈ *set_tree l* ∪ *set_tree r*. *m* ≤ *x*) ∧ *heap l* ∧ *heap r*)

fun *ltree* :: *'a lheap* ⇒ *bool* **where**
ltree Leaf = *True* |
ltree (*Node l* (*a*, *n*) *r*) =
 (*min_height l* ≥ *min_height r* ∧ *n* = *min_height r* + *1* ∧ *ltree l* & *ltree r*)

definition *empty* :: *'a lheap* **where**
empty = *Leaf*

definition *node* :: *'a lheap* ⇒ *'a* ⇒ *'a lheap* ⇒ *'a lheap* **where**
node l a r =
 (*let mhl* = *mht l*; *mhr* = *mht r*
  in if mhl ≥ *mhr then Node l* (*a*,*mhr*+*1*) *r else Node r* (*a*,*mhl*+*1*) *l*)

fun *get_min* :: *'a lheap* ⇒ *'a* **where**
get_min(*Node l* (*a*, *n*) *r*) = *a*

For function *merge*:

unbundle *pattern_aliases*

fun *merge* :: *'a*::*ord lheap* ⇒ *'a lheap* ⇒ *'a lheap* **where**
merge Leaf t = *t* |
merge t Leaf = *t* |
merge (*Node l1* (*a1*, *n1*) *r1* =: *t1*) (*Node l2* (*a2*, *n2*) *r2* =: *t2*) =
   (*if a1* ≤ *a2 then node l1 a1* (*merge r1 t2*)
    else node l2 a2 (*merge t1 r2*))

Termination of *merge*: by sum or lexicographic product of the sizes of the two arguments. Isabelle uses a lexicographic product.

lemma *merge_code*: *merge t1 t2* = (*case* (*t1*,*t2*) *of*
  (*Leaf*, _) ⇒ *t2* |
  (_, *Leaf*) ⇒ *t1* |
  (*Node l1* (*a1*, *n1*) *r1*, *Node l2* (*a2*, *n2*) *r2*) ⇒
    if a1 ≤ *a2 then node l1 a1* (*merge r1 t2*) *else node l2 a2* (*merge t1 r2*))
by(*induction t1 t2 rule*: *merge.induct*) (*simp_all split*: *tree.split*)

hide_const (**open**) *insert*

definition *insert* :: *'a*::*ord* ⇒ *'a lheap* ⇒ *'a lheap* **where**

insert x t = merge (Node Leaf (x,1) Leaf) t

fun *del_min* :: *'a::ord lheap ⇒ 'a lheap* **where**
del_min Leaf = Leaf |
del_min (Node l _ r) = merge l r

47.1 Lemmas

lemma *mset_tree_empty*: *mset_tree t = {#} ⟷ t = Leaf*
by(*cases t*) *auto*

lemma *mht_eq_min_height*: *ltree t ⟹ mht t = min_height t*
by(*cases t*) *auto*

lemma *ltree_node*: *ltree (node l a r) ⟷ ltree l ∧ ltree r*
by(*auto simp add*: *node_def mht_eq_min_height*)

lemma *heap_node*: *heap (node l a r) ⟷*
heap l ∧ heap r ∧ (∀ x ∈ set_tree l ∪ set_tree r. a ≤ x)
by(*auto simp add*: *node_def*)

lemma *set_tree_mset*: *set_tree t = set_mset(mset_tree t)*
by(*induction t*) *auto*

47.2 Functional Correctness

lemma *mset_merge*: *mset_tree (merge t1 t2) = mset_tree t1 + mset_tree t2*
by (*induction t1 t2 rule*: *merge.induct*) (*auto simp add*: *node_def ac_simps*)

lemma *mset_insert*: *mset_tree (insert x t) = mset_tree t + {#x#}*
by (*auto simp add*: *insert_def mset_merge*)

lemma *get_min*: ⟦ *heap t*; *t ≠ Leaf* ⟧ ⟹ *get_min t = Min(set_tree t)*
by (*cases t*) (*auto simp add*: *eq_Min_iff*)

lemma *mset_del_min*: *mset_tree (del_min t) = mset_tree t − {# get_min t #}*
by (*cases t*) (*auto simp*: *mset_merge*)

lemma *ltree_merge*: ⟦ *ltree l*; *ltree r* ⟧ ⟹ *ltree (merge l r)*
by(*induction l r rule*: *merge.induct*)(*auto simp*: *ltree_node*)

lemma *heap_merge*: ⟦ *heap l*; *heap r* ⟧ ⟹ *heap (merge l r)*

proof(*induction l r rule*: *merge.induct*)
case *3* **thus** *?case* **by**(*auto simp*: *heap_node mset_merge ball_Un set_tree_mset*)
qed *simp_all*

lemma *ltree_insert*: *ltree t* $\Longrightarrow$ *ltree*(*insert x t*)
by(*simp add*: *insert_def ltree_merge del*: *merge.simps split*: *tree.split*)

lemma *heap_insert*: *heap t* $\Longrightarrow$ *heap*(*insert x t*)
by(*simp add*: *insert_def heap_merge del*: *merge.simps split*: *tree.split*)

lemma *ltree_del_min*: *ltree t* $\Longrightarrow$ *ltree*(*del_min t*)
by(*cases t*)(*auto simp add*: *ltree_merge simp del*: *merge.simps*)

lemma *heap_del_min*: *heap t* $\Longrightarrow$ *heap*(*del_min t*)
by(*cases t*)(*auto simp add*: *heap_merge simp del*: *merge.simps*)

Last step of functional correctness proof: combine all the above lemmas to show that leftist heaps satisfy the specification of priority queues with merge.

interpretation *lheap*: *Priority_Queue_Merge*
where *empty* = *empty* **and** *is_empty* = $\lambda t.$ *t* = *Leaf*
and *insert* = *insert* **and** *del_min* = *del_min*
and *get_min* = *get_min* **and** *merge* = *merge*
and *invar* = $\lambda t.$ *heap t* $\wedge$ *ltree t* **and** *mset* = *mset_tree*
proof(*standard*, *goal_cases*)
  case *1* **show** *?case* **by** (*simp add*: *empty_def*)
next
  case (*2 q*) **show** *?case* **by** (*cases q*) *auto*
next
  case *3* **show** *?case* **by**(*rule mset_insert*)
next
  case *4* **show** *?case* **by**(*rule mset_del_min*)
next
  case *5* **thus** *?case* **by**(*simp add*: *get_min mset_tree_empty set_tree_mset*)
next
  case *6* **thus** *?case* **by**(*simp add*: *empty_def*)
next
  case *7* **thus** *?case* **by**(*simp add*: *heap_insert ltree_insert*)
next
  case *8* **thus** *?case* **by**(*simp add*: *heap_del_min ltree_del_min*)
next
  case *9* **thus** *?case* **by** (*simp add*: *mset_merge*)
next
  case *10* **thus** *?case* **by** (*simp add*: *heap_merge ltree_merge*)

qed

47.3 Complexity

We count only the calls of the only recursive function: *merge*

Explicit termination argument: sum of sizes

```
fun T_merge :: 'a::ord lheap ⇒ 'a lheap ⇒ nat where
T_merge Leaf t = 1 |
T_merge t Leaf = 1 |
T_merge (Node l1 (a1, n1) r1 =: t1) (Node l2 (a2, n2) r2 =: t2) =
  (if a1 ≤ a2 then T_merge r1 t2
   else T_merge t1 r2) + 1

definition T_insert :: 'a::ord ⇒ 'a lheap ⇒ nat where
T_insert x t = T_merge (Node Leaf (x, 1) Leaf) t

fun T_del_min :: 'a::ord lheap ⇒ nat where
T_del_min Leaf = 0 |
T_del_min (Node l _ r) = T_merge l r

lemma T_merge_min_height: ltree l ⟹ ltree r ⟹ T_merge l r ≤ min_height
l + min_height r + 1
proof(induction l r rule: merge.induct)
  case 3 thus ?case by(auto)
qed simp_all

corollary T_merge_log: assumes ltree l ltree r
  shows T_merge l r ≤ log 2 (size1 l) + log 2 (size1 r) + 1
using le_log2_of_power[OF min_height_size1[of l]]
  le_log2_of_power[OF min_height_size1[of r]] T_merge_min_height[of
l r] assms
by linarith

corollary T_insert_log: ltree t ⟹ T_insert x t ≤ log 2 (size1 t) + 2
using T_merge_log[of Node Leaf (x, 1) Leaf t]
by(simp add: T_insert_def split: tree.split)


lemma ld_ld_1_less:
  assumes x > 0 y > 0 shows log 2 x + log 2 y + 1 < 2 * log 2 (x+y)
proof -
  have 2 powr (log 2 x + log 2 y + 1) = 2*x*y
    using assms by(simp add: powr_add)
```

also have ... < (*x*+*y*)^*2* **using** *assms*
by(*simp add*: *numeral_eq_Suc algebra_simps add_pos_pos*)
also have ... = *2 powr* (*2* ∗ *log 2* (*x*+*y*))
using *assms* **by**(*simp add*: *powr_add log_powr*[*symmetric*])
finally show *?thesis* **by** *simp*
qed

corollary *T_del_min_log*: **assumes** *ltree t*
shows *T_del_min t* ≤ *2* ∗ *log 2* (*size1 t*)
proof(*cases t rule*: *tree2_cases*)
case *Leaf* **thus** *?thesis* **using** *assms* **by** *simp*
next
case [*simp*]: (*Node l _ _ r*)
have *T_del_min t* = *T_merge l r* **by** *simp*
also have ... ≤ *log 2* (*size1 l*) + *log 2* (*size1 r*) + *1*
using ‹*ltree t*› *T_merge_log*[*of l r*] **by** (*auto simp del*: *T_merge.simps*)
also have ... ≤ *2* ∗ *log 2* (*size1 t*)
using *ld_ld_1_less*[*of size1 l size1 r*] **by** (*simp*)
finally show *?thesis* **.**
qed

end

theory *Leftist_Heap_List*
imports
Leftist_Heap
Complex_Main
begin

47.4 Converting a list into a leftist heap

fun *merge_adj* :: (′*a*::*ord*) *lheap list* ⇒ ′*a lheap list* **where**
merge_adj [] = [] |
merge_adj [*t*] = [*t*] |
merge_adj (*t1* # *t2* # *ts*) = *merge t1 t2* # *merge_adj ts*

For the termination proof of *merge_all* below.

lemma *length_merge_adjacent*[*simp*]: *length* (*merge_adj ts*) = (*length ts* + *1*) *div 2*
by (*induction ts rule*: *merge_adj.induct*) *auto*

fun *merge_all* :: (′*a*::*ord*) *lheap list* ⇒ ′*a lheap* **where**
merge_all [] = *Leaf* |

merge_all [*t*] = *t* |
merge_all ts = *merge_all* (*merge_adj ts*)

47.4.1 Functional correctness

lemma *heap_merge_adj*: ∀ *t* ∈ *set ts*. *heap t* ⟹ ∀ *t* ∈ *set* (*merge_adj ts*). *heap t*
by(*induction ts rule*: *merge_adj.induct*) (*auto simp*: *heap_merge*)

lemma *ltree_merge_adj*: ∀ *t* ∈ *set ts*. *ltree t* ⟹ ∀ *t* ∈ *set* (*merge_adj ts*). *ltree t*
by(*induction ts rule*: *merge_adj.induct*) (*auto simp*: *ltree_merge*)

lemma *heap_merge_all*: ∀ *t* ∈ *set ts*. *heap t* ⟹ *heap* (*merge_all ts*)
apply(*induction ts rule*: *merge_all.induct*)
using [[*simp_depth_limit=3*]] **by** (*auto simp add*: *heap_merge_adj*)

lemma *ltree_merge_all*: ∀ *t* ∈ *set ts*. *ltree t* ⟹ *ltree* (*merge_all ts*)
apply(*induction ts rule*: *merge_all.induct*)
using [[*simp_depth_limit=3*]] **by** (*auto simp add*: *ltree_merge_adj*)

lemma *mset_merge_adj*:
$\sum_{\#}$ (*image_mset mset_tree* (*mset* (*merge_adj ts*))) =
$\sum_{\#}$ (*image_mset mset_tree* (*mset ts*))
by(*induction ts rule*: *merge_adj.induct*) (*auto simp*: *mset_merge*)

lemma *mset_merge_all*:
mset_tree (*merge_all ts*) = $\sum_{\#}$ (*mset* (*map mset_tree ts*))
by(*induction ts rule*: *merge_all.induct*) (*auto simp*: *mset_merge mset_merge_adj*)

fun *lheap_list* :: *'a*::*ord list* ⇒ *'a lheap* **where**
lheap_list xs = *merge_all* (*map* (λ*x*. *Node Leaf* (*x*,*1*) *Leaf*) *xs*)

lemma *mset_lheap_list*: *mset_tree* (*lheap_list xs*) = *mset xs*
by (*simp add*: *mset_merge_all o_def*)

lemma *ltree_lheap_list*: *ltree* (*lheap_list ts*)
by(*simp add*: *ltree_merge_all*)

lemma *heap_lheap_list*: *heap* (*lheap_list ts*)
by(*simp add*: *heap_merge_all*)

lemma *size_merge*: *size*(*merge t1 t2*) = *size t1* + *size t2*
by(*induction t1 t2 rule*: *merge.induct*) (*auto simp*: *node_def*)

47.4.2 Running time

```
fun T_merge_adj :: ('a::ord) lheap list ⇒ nat where
T_merge_adj [] = 0 |
T_merge_adj [t] = 0 |
T_merge_adj (t1 # t2 # ts) = T_merge t1 t2 + T_merge_adj ts

fun T_merge_all :: ('a::ord) lheap list  ⇒ nat where
T_merge_all [] = 0 |
T_merge_all [t] = 0 |
T_merge_all ts = T_merge_adj ts + T_merge_all (merge_adj ts)

fun T_lheap_list :: 'a::ord list ⇒ nat where
T_lheap_list xs = T_merge_all (map (λx. Node Leaf (x,1) Leaf) xs)

abbreviation Tm where
Tm n == 2 * log 2 (n+1) + 1

lemma T_merge_adj: ⟦ ∀ t ∈ set ts. ltree t; ∀ t ∈ set ts. size t = n ⟧
  ⟹ T_merge_adj ts ≤ (length ts div 2) * Tm n
proof(induction ts rule: T_merge_adj.induct)
  case 1 thus ?case by simp
next
  case 2 thus ?case by simp
next
   case (3 t1 t2) thus ?case using T_merge_log[of t1 t2] by (simp add:
algebra_simps size1_size)
qed

lemma size_merge_adj:
  ⟦ even(length ts); ∀ t ∈ set ts. ltree t; ∀ t ∈ set ts. size t = n ⟧
   ⟹ ∀ t ∈ set (merge_adj ts). size t = 2*n
by(induction ts rule: merge_adj.induct) (auto simp: size_merge)

lemma T_merge_all:
 ⟦ ∀ t ∈ set ts. ltree t; ∀ t ∈ set ts. size t = n; length ts = 2^k ⟧
  ⟹ T_merge_all ts ≤ (∑ i=1..k. 2^(k−i) * Tm(2 ^ (i−1) * n))
proof (induction ts arbitrary: k n rule: merge_all.induct)
  case 1 thus ?case by simp
next
  case 2 thus ?case by simp
next
  case (3 t1 t2 ts)
  let ?ts = t1 # t2 # ts
```

```
  let ?ts2 = merge_adj ?ts
  obtain k' where k': k = Suc k' using 3.prems(3)
   by (metis length_Cons nat.inject nat_power_eq_Suc_0_iff nat.exhaust)
  have 1: ∀ x ∈ set(merge_adj ?ts). ltree x
    by(rule ltree_merge_adj[OF 3.prems(1)])
  have even (length ts) using 3.prems(3) even_Suc_Suc_iff by fastforce
  from 3.prems(2) size_merge_adj[OF this] 3.prems(1)
  have 2: ∀ x ∈ set(merge_adj ?ts). size x = 2*n by(auto simp: size_merge)
  have 3: length ?ts2 = 2 ^ k' using 3.prems(3) k' by auto
  have 4: length ?ts div 2 = 2 ^ k'
    using 3.prems(3) k' by(simp add: power_eq_if[of 2 k] split: if_splits)
  have T_merge_all ?ts = T_merge_adj ?ts + T_merge_all ?ts2 by simp
  also have ... ≤ 2^k' * Tm n + T_merge_all ?ts2
    using 4 T_merge_adj[OF 3.prems(1,2)] by auto
  also have ... ≤ 2^k' * Tm n + (∑ i=1..k'. 2^(k'−i) * Tm(2 ^ (i−1) *
(2*n)))
    using 3.IH[OF 1 2 3] by simp
  also have ... = 2^k' * Tm n + (∑ i=1..k'. 2^(k'−i) * Tm(2 ^ (Suc(i−1))
* n))
    by (simp add: mult_ac cong del: sum.cong)
  also have ... = 2^k' * Tm n + (∑ i=1..k'. 2^(k'−i) * Tm(2 ^ i * n))
     by (simp)
  also have ... = (∑ i=1..k. 2^(k−i) * Tm(2 ^ (i−1) * real n))
   by(simp add: sum.atLeast_Suc_atMost[of Suc 0 Suc k'] sum.atLeast_Suc_atMost_Suc_shift[of
_ Suc 0] k'
        del: sum.cl_ivl_Suc)
  finally show ?case .
qed

lemma summation: (∑ i=1..k. 2^(k−i) * ((2::real)*i+1)) = 5*2^k −
(2::real)*k − 5
proof (induction k)
  case 0 thus ?case by simp
next
  case (Suc k)
  have (∑ i=1..Suc k. 2^(Suc k − i) * ((2::real)*i+1))
    = (∑ i=1..k. 2^(k+1−i) * ((2::real)*i+1)) + 2*k+3
    by(simp)
   also have ... = (∑ i=1..k. (2::real)*(2^(k−i) * ((2::real)*i+1))) +
2*k+3
    by (simp add: Suc_diff_le mult.assoc)
  also have ... = 2*(∑ i=1..k. 2^(k−i) * ((2::real)*i+1)) + 2*k+3
    by(simp add: sum_distrib_left)
  also have ... = (2::real)*(5*2^k − (2::real)*k − 5) + 2*k+3
```

```
    using Suc.IH by simp
  also have ... = 5*2^(Suc k) − (2::real)*(Suc k) − 5
    by simp
  finally show ?case .
qed

lemma T_lheap_list: assumes length xs = 2 ^ k
shows T_lheap_list xs ≤ 5 * length xs
proof −
  let ?ts = map (λx. Node Leaf (x,1) Leaf) xs
  have T_lheap_list xs = T_merge_all ?ts by simp
  also have ... ≤ (∑ i = 1..k. 2^(k−i) * (2 * log 2 (2^(i−1) + 1) + 1))
    using T_merge_all[of ?ts 1 k] assms by (simp)
  also have ... ≤ (∑ i = 1..k. 2^(k−i) * (2 * log 2 (2*2^(i−1)) + 1))
    apply(rule sum_mono)
    using zero_le_power[of 2::real] by (simp add: add_pos_nonneg)
  also have ... = (∑ i = 1..k. 2^(k−i) * (2 * log 2 (2^(1+(i−1))) + 1))
    by (simp del: Suc_pred)
  also have ... = (∑ i = 1..k. 2^(k−i) * (2 * log 2 (2^i) + 1))
    by (simp)
  also have ... = (∑ i = 1..k. 2^(k−i) * ((2::real)*i+1))
    by (simp add:log_nat_power algebra_simps)
  also have ... = 5*(2::real)^k − (2::real)*k − 5
    using summation by (simp)
  also have ... ≤ 5*(2::real)^k
    by linarith
  finally show ?thesis
    using assms of_nat_le_iff by fastforce
qed

end
```

48 Binomial Heap

```
theory Binomial_Heap
imports
  HOL−Library.Pattern_Aliases
  Complex_Main
  Priority_Queue_Specs
begin
```

We formalize the binomial heap presentation from Okasaki's book. We show the functional correctness and complexity of all operations.

The presentation is engineered for simplicity, and most proofs are straightforward and automatic.

48.1 Binomial Tree and Heap Datatype

datatype *'a tree* = *Node* (*rank*: *nat*) (*root*: *'a*) (*children*: *'a tree list*)

type_synonym *'a trees* = *'a tree list*

48.1.1 Multiset of elements

fun *mset_tree* :: *'a*::*linorder tree* ⇒ *'a multiset* **where**
  mset_tree (*Node* _ *a ts*) = {#*a*#} + (∑ *t*∈#*mset ts*. *mset_tree t*)

definition *mset_trees* :: *'a*::*linorder trees* ⇒ *'a multiset* **where**
  mset_trees ts = (∑ *t*∈#*mset ts*. *mset_tree t*)

lemma *mset_tree_simp_alt*[*simp*]:
  mset_tree (*Node r a ts*) = {#*a*#} + *mset_trees ts*
  unfolding *mset_trees_def* **by** *auto*
declare *mset_tree.simps*[*simp del*]

lemma *mset_tree_nonempty*[*simp*]: *mset_tree t* ≠ {#}
by (*cases t*) *auto*

lemma *mset_trees_Nil*[*simp*]:
  mset_trees [] = {#}
by (*auto simp*: *mset_trees_def*)

lemma *mset_trees_Cons*[*simp*]: *mset_trees* (*t*#*ts*) = *mset_tree t* + *mset_trees ts*
by (*auto simp*: *mset_trees_def*)

lemma *mset_trees_empty_iff*[*simp*]: *mset_trees ts* = {#} ⟷ *ts*=[]
by (*auto simp*: *mset_trees_def*)

lemma *root_in_mset*[*simp*]: *root t* ∈# *mset_tree t*
by (*cases t*) *auto*

lemma *mset_trees_rev_eq*[*simp*]: *mset_trees* (*rev ts*) = *mset_trees ts*
by (*auto simp*: *mset_trees_def*)

48.1.2 Invariants

Binomial tree

fun *btree* :: *'a::linorder tree* ⇒ *bool* **where**
btree (*Node r x ts*) ⟷
  (∀ *t*∈*set ts*. *btree t*) ∧ *map rank ts* = *rev* [*0*..<*r*]

Heap invariant

fun *heap* :: *'a::linorder tree* ⇒ *bool* **where**
heap (*Node _ x ts*) ⟷ (∀ *t*∈*set ts*. *heap t* ∧ *x* ≤ *root t*)

definition *bheap t* ⟷ *btree t* ∧ *heap t*

Binomial Heap invariant

definition *invar ts* ⟷ (∀ *t*∈*set ts*. *bheap t*) ∧ (*sorted_wrt* (<) (*map rank ts*))

The children of a node are a valid heap

lemma *invar_children*:
  bheap (*Node r v ts*) ⟹ *invar* (*rev ts*)
  by (*auto simp*: *bheap_def invar_def rev_map*[*symmetric*])

48.2 Operations and Their Functional Correctness

48.2.1 *link*

context
includes *pattern_aliases*
begin

fun *link* :: (*'a::linorder*) *tree* ⇒ *'a tree* ⇒ *'a tree* **where**
  link (*Node r* x_1 ts_1 =: t_1) (*Node r'* x_2 ts_2 =: t_2) =
    (*if* $x_1 \leq x_2$ *then Node* (*r*+*1*) x_1 (t_2#ts_1) *else Node* (*r*+*1*) x_2 (t_1#ts_2))

end

lemma *invar_link*:
  assumes *bheap* t_1
  assumes *bheap* t_2
  assumes *rank* t_1 = *rank* t_2
  shows *bheap* (*link* t_1 t_2)
using *assms* **unfolding** *bheap_def*
by (*cases* (t_1, t_2) *rule*: *link.cases*) *auto*

lemma *rank_link*[*simp*]: *rank* (*link* t_1 t_2) = *rank* t_1 + *1*
by (*cases* (t_1, t_2) *rule*: *link.cases*) *simp*

lemma *mset_link*[*simp*]: *mset_tree* (*link* t_1 t_2) = *mset_tree* t_1 + *mset_tree* t_2
by (*cases* (t_1, t_2) *rule*: *link.cases*) *simp*

48.2.2 *ins_tree*

fun *ins_tree* :: *'a::linorder tree* ⇒ *'a trees* ⇒ *'a trees* **where**
ins_tree t [] = [*t*]
| *ins_tree* t_1 (t_2#*ts*) =
(*if rank* t_1 < *rank* t_2 *then* t_1#t_2#*ts else ins_tree* (*link* t_1 t_2) *ts*)

lemma *bheap0*[*simp*]: *bheap* (*Node 0 x* [])
unfolding *bheap_def* **by** *auto*

lemma *invar_Cons*[*simp*]:
invar (*t*#*ts*)
⟷ *bheap t* ∧ *invar ts* ∧ (∀ *t'*∈*set ts. rank t* < *rank t'*)
by (*auto simp*: *invar_def*)

lemma *invar_ins_tree*:
assumes *bheap t*
assumes *invar ts*
assumes ∀ *t'*∈*set ts. rank t* ≤ *rank t'*
shows *invar* (*ins_tree t ts*)
using *assms*
by (*induction t ts rule*: *ins_tree.induct*) (*auto simp*: *invar_link less_eq_Suc_le*[*symmetric*])

lemma *mset_trees_ins_tree*[*simp*]:
mset_trees (*ins_tree t ts*) = *mset_tree t* + *mset_trees ts*
by (*induction t ts rule*: *ins_tree.induct*) *auto*

lemma *ins_tree_rank_bound*:
assumes *t'* ∈ *set* (*ins_tree t ts*)
assumes ∀ *t'*∈*set ts. rank* t_0 < *rank t'*
assumes *rank* t_0 < *rank t*
shows *rank* t_0 < *rank t'*
using *assms*
by (*induction t ts rule*: *ins_tree.induct*) (*auto split*: *if_splits*)

48.2.3 *insert*

hide_const (**open**) *insert*

definition *insert* :: *'a::linorder* ⇒ *'a trees* ⇒ *'a trees* **where**

insert x ts = *ins_tree* (*Node 0 x* []) *ts*

lemma *invar_insert*[*simp*]: *invar t* ⟹ *invar* (*insert x t*)
by (*auto intro*!: *invar_ins_tree simp*: *insert_def*)

lemma *mset_trees_insert*[*simp*]: *mset_trees* (*insert x t*) = {#*x*#} + *mset_trees*
t
by(*auto simp*: *insert_def*)

48.2.4 *merge*

context
includes *pattern_aliases*
begin

fun *merge* :: *'a::linorder trees* ⇒ *'a trees* ⇒ *'a trees* **where**
  merge ts_1 [] = ts_1
| *merge* [] ts_2 = ts_2
| *merge* (t_1#ts_1 =: h_1) (t_2#ts_2 =: h_2) = (
    if rank t_1 < *rank* t_2 *then* t_1 # *merge* ts_1 h_2 *else*
    if rank t_2 < *rank* t_1 *then* t_2 # *merge* h_1 ts_2
    else ins_tree (*link* t_1 t_2) (*merge* ts_1 ts_2)
)

end

lemma *merge_simp2*[*simp*]: *merge* [] ts_2 = ts_2
by (*cases* ts_2) *auto*

lemma *merge_rank_bound*:
  assumes $t' \in$ *set* (*merge* ts_1 ts_2)
  assumes ∀ t_1∈*set* ts_1. *rank t* < *rank* t_1
  assumes ∀ t_2∈*set* ts_2. *rank t* < *rank* t_2
  shows *rank t* < *rank* t'
using *assms*
by (*induction* ts_1 ts_2 *arbitrary*: t' *rule*: *merge.induct*)
  (*auto split*: *if_splits simp*: *ins_tree_rank_bound*)

lemma *invar_merge*[*simp*]:
  assumes *invar* ts_1
  assumes *invar* ts_2
  shows *invar* (*merge* ts_1 ts_2)
using *assms*
by (*induction* ts_1 ts_2 *rule*: *merge.induct*)

(*auto 0 3 simp*: *Suc_le_eq intro*!: *invar_ins_tree invar_link elim*!: *merge_rank_bound*)

Longer, more explicit proof of *invar_merge*, to illustrate the application of the *merge_rank_bound* lemma.

```
lemma
  assumes invar ts₁
  assumes invar ts₂
  shows invar (merge ts₁ ts₂)
  using assms
proof (induction ts₁ ts₂ rule: merge.induct)
  case (3 t₁ ts₁ t₂ ts₂)
  — Invariants of the parts can be shown automatically
  from 3.prems have [simp]:
    bheap t₁ bheap t₂

    by auto

  — These are the three cases of the merge function
  consider (LT) rank t₁ < rank t₂
       | (GT) rank t₁ > rank t₂
       | (EQ) rank t₁ = rank t₂
    using antisym_conv3 by blast
  then show ?case proof cases
    case LT
    — merge takes the first tree from the left heap
    then have merge (t₁ # ts₁) (t₂ # ts₂) = t₁ # merge ts₁ (t₂ # ts₂) by
simp
    also have invar ... proof (simp, intro conjI)
      — Invariant follows from induction hypothesis
      show invar (merge ts₁ (t₂ # ts₂))
        using LT 3.IH 3.prems by simp

      — It remains to show that t₁ has smallest rank.
      show ∀t'∈set (merge ts₁ (t₂ # ts₂)). rank t₁ < rank t'
        — Which is done by auxiliary lemma merge_rank_bound
        using LT 3.prems by (force elim!: merge_rank_bound)
    qed
    finally show ?thesis .
  next
    — merge takes the first tree from the right heap
    case GT
    — The proof is anaologous to the LT case
   then show ?thesis using 3.prems 3.IH by (force elim!: merge_rank_bound)
  next
```

case [*simp*]: *EQ*
— *merge* links both first trees, and inserts them into the merged remaining heaps
have *merge* (t_1 # ts_1) (t_2 # ts_2) = *ins_tree* (*link* t_1 t_2) (*merge* ts_1 ts_2) **by** *simp*
also have *invar* ... **proof** (*intro invar_ins_tree invar_link*)
— Invariant of merged remaining heaps follows by IH
show *invar* (*merge* ts_1 ts_2)
using *EQ 3.prems 3.IH* **by** *auto*

— For insertion, we have to show that the rank of the linked tree is ≤ the ranks in the merged remaining heaps
show ∀ *t'*∈*set* (*merge* ts_1 ts_2). *rank* (*link* t_1 t_2) ≤ *rank* *t'*
proof −
— Which is, again, done with the help of *merge_rank_bound*
have *rank* (*link* t_1 t_2) = *Suc* (*rank* t_2) **by** *simp*
thus *?thesis* **using** *3.prems* **by** (*auto simp*: *Suc_le_eq elim!*: *merge_rank_bound*)
qed
qed *simp_all*
finally show *?thesis* **.**
qed
qed *auto*

lemma *mset_trees_merge*[*simp*]:
mset_trees (*merge* ts_1 ts_2) = *mset_trees* ts_1 + *mset_trees* ts_2
by (*induction* ts_1 ts_2 *rule*: *merge.induct*) *auto*

48.2.5 *get_min*

fun *get_min* :: *'a*::*linorder trees* ⇒ *'a* **where**
get_min [*t*] = *root t*
| *get_min* (*t*#*ts*) = *min* (*root t*) (*get_min ts*)

lemma *bheap_root_min*:
assumes *bheap t*
assumes *x* ∈# *mset_tree t*
shows *root t* ≤ *x*
using *assms* **unfolding** *bheap_def*
by (*induction t arbitrary*: *x rule*: *mset_tree.induct*) (*fastforce simp*: *mset_trees_def*)

lemma *get_min_mset*:
assumes *ts*≠[]

```
  assumes invar ts
  assumes x ∈# mset_trees ts
  shows get_min ts ≤ x
  using assms
apply (induction ts arbitrary: x rule: get_min.induct)
apply (auto
      simp: bheap_root_min min_def intro: order_trans;
      meson linear order_trans bheap_root_min
      )+
done

lemma get_min_member:
  ts≠[] ⟹ get_min ts ∈# mset_trees ts
by (induction ts rule: get_min.induct) (auto simp: min_def)

lemma get_min:
  assumes mset_trees ts ≠ {#}
  assumes invar ts
  shows get_min ts = Min_mset (mset_trees ts)
using assms get_min_member get_min_mset
by (auto simp: eq_Min_iff)
```

48.2.6 *get_min_rest*

```
fun get_min_rest :: 'a::linorder trees ⇒ 'a tree × 'a trees where
  get_min_rest [t] = (t,[])
| get_min_rest (t#ts) = (let (t',ts') = get_min_rest ts
                   in if root t ≤ root t' then (t,ts) else (t',t#ts'))

lemma get_min_rest_get_min_same_root:
  assumes ts≠[]
  assumes get_min_rest ts = (t',ts')
  shows root t' = get_min ts
using assms
by (induction ts arbitrary: t' ts' rule: get_min.induct) (auto simp: min_def
split: prod.splits)

lemma mset_get_min_rest:
  assumes get_min_rest ts = (t',ts')
  assumes ts≠[]
  shows mset ts = {#t'#} + mset ts'
using assms
by (induction ts arbitrary: t' ts' rule: get_min.induct) (auto split: prod.splits
if_splits)
```

lemma *set_get_min_rest*:
  assumes *get_min_rest ts* = (*t'*, *ts'*)
  assumes *ts*≠[]
  shows *set ts* = *Set.insert t'* (*set ts'*)
using *mset_get_min_rest*[*OF assms*, *THEN arg_cong*[**where** *f*=*set_mset*]]
by *auto*

lemma *invar_get_min_rest*:
  assumes *get_min_rest ts* = (*t'*,*ts'*)
  assumes *ts*≠[]
  assumes *invar ts*
  shows *bheap t'* **and** *invar ts'*
proof −
  have *bheap t'* ∧ *invar ts'*
    using *assms*
    proof (*induction ts arbitrary*: *t' ts' rule*: *get_min.induct*)
      case (*2 t v va*)
      then show *?case*
        apply (*clarsimp split*: *prod.splits if_splits*)
        apply (*drule set_get_min_rest*; *fastforce*)
        done
    qed *auto*
  thus *bheap t'* **and** *invar ts'* **by** *auto*
qed

48.2.7 *del_min*

definition *del_min* :: *'a*::*linorder trees* ⇒ *'a*::*linorder trees* **where**
del_min ts = (*case get_min_rest ts of*
  (*Node r x* ts_1, ts_2) ⇒ *merge* (*rev* ts_1) ts_2)

lemma *invar_del_min*[*simp*]:
  assumes *ts* ≠ []
  assumes *invar ts*
  shows *invar* (*del_min ts*)
using *assms*
unfolding *del_min_def*
by (*auto*
    split: *prod.split tree.split*
    intro!: *invar_merge invar_children*
    dest: *invar_get_min_rest*
)

lemma *mset_trees_del_min*:
  assumes *ts* ≠ []
  shows *mset_trees ts* = *mset_trees* (*del_min ts*) + {# *get_min ts* #}
using *assms*
unfolding *del_min_def*
apply (*clarsimp split*: *tree.split prod.split*)
apply (*frule* (*1*) *get_min_rest_get_min_same_root*)
apply (*frule* (*1*) *mset_get_min_rest*)
apply (*auto simp*: *mset_trees_def*)
done

48.2.8 Instantiating the Priority Queue Locale

Last step of functional correctness proof: combine all the above lemmas to show that binomial heaps satisfy the specification of priority queues with merge.

interpretation *bheaps*: *Priority_Queue_Merge*
  where *empty* = [] **and** *is_empty* = (=) [] **and** *insert* = *insert*
  and *get_min* = *get_min* **and** *del_min* = *del_min* **and** *merge* = *merge*
  and *invar* = *invar* **and** *mset* = *mset_trees*
proof (*unfold_locales*, *goal_cases*)
  case *1* **thus** *?case* **by** *simp*
next
  case *2* **thus** *?case* **by** *auto*
next
  case *3* **thus** *?case* **by** *auto*
next
  case (*4 q*)
  thus *?case* **using** *mset_trees_del_min*[*of q*] *get_min*[*OF* _ ‹*invar q*›]
    by (*auto simp*: *union_single_eq_diff*)
next
  case (*5 q*) **thus** *?case* **using** *get_min*[*of q*] **by** *auto*
next
  case *6* **thus** *?case* **by** (*auto simp add*: *invar_def*)
next
  case *7* **thus** *?case* **by** *simp*
next
  case *8* **thus** *?case* **by** *simp*
next
  case *9* **thus** *?case* **by** *simp*
next
  case *10* **thus** *?case* **by** *simp*
qed

48.3 Complexity

The size of a binomial tree is determined by its rank

```
lemma size_mset_btree:
  assumes btree t
  shows size (mset_tree t) = 2^rank t
  using assms
proof (induction t)
  case (Node r v ts)
  hence IH: size (mset_tree t) = 2^rank t if t ∈ set ts for t
    using that by auto

  from Node have COMPL: map rank ts = rev [0..<r] by auto

  have size (mset_trees ts) = (∑t←ts. size (mset_tree t))
    by (induction ts) auto
  also have ... = (∑t←ts. 2^rank t) using IH
    by (auto cong: map_cong)
  also have ... = (∑r←map rank ts. 2^r)
    by (induction ts) auto
  also have ... = (∑i∈{0..<r}. 2^i)
    unfolding COMPL
   by (auto simp: rev_map[symmetric] interv_sum_list_conv_sum_set_nat)
  also have ... = 2^r - 1
    by (induction r) auto
  finally show ?case
    by (simp)
qed

lemma size_mset_tree:
  assumes bheap t
  shows size (mset_tree t) = 2^rank t
using assms unfolding bheap_def
by (simp add: size_mset_btree)
```

The length of a binomial heap is bounded by the number of its elements

```
lemma size_mset_trees:
  assumes invar ts
  shows length ts ≤ log 2 (size (mset_trees ts) + 1)
proof -
  from ‹invar ts› have
    ASC: sorted_wrt (<) (map rank ts) and
    TINV: ∀t∈set ts. bheap t
    unfolding invar_def by auto
```

have *(2::nat)^length ts* = ($\sum$ *i*∈{*0..<length ts*}. *2^i*) + *1*
by (*simp add*: *sum_power2*)
also have ... = ($\sum$ *i*←[*0..<length ts*]. *2^i*) + *1* (**is** _ = *?S* + *1*)
by (*simp add*: *interv_sum_list_conv_sum_set_nat*)
also have *?S* ≤ ($\sum$ *t*←*ts*. *2^rank t*) (**is** _ ≤ *?T*)
using *sorted_wrt_less_idx*[*OF ASC*] **by**(*simp add*: *sum_list_mono2*)
also have *?T* + *1* ≤ ($\sum$ *t*←*ts*. *size* (*mset_tree t*)) + *1* **using** *TINV*
by (*auto cong*: *map_cong simp*: *size_mset_tree*)
also have ... = *size* (*mset_trees ts*) + *1*
unfolding *mset_trees_def* **by** (*induction ts*) *auto*
finally have *2^length ts* ≤ *size* (*mset_trees ts*) + *1* **by** *simp*
then show *?thesis* **using** *le_log2_of_power* **by** *blast*
qed

48.3.1 Timing Functions

We define timing functions for each operation, and provide estimations of their complexity.

definition *T_link* :: *'a::linorder tree* ⇒ *'a tree* ⇒ *nat* **where**
[*simp*]: *T_link* _ _ = *1*

This function is non-canonical: we omitted a +*1* in the *else*-part, to keep the following analysis simpler and more to the point.

fun *T_ins_tree* :: *'a::linorder tree* ⇒ *'a trees* ⇒ *nat* **where**
T_ins_tree t [] = *1*
| *T_ins_tree* t_1 (t_2 # *ts*) = (
(*if rank* t_1 < *rank* t_2 *then 1*
else T_link t_1 t_2 + *T_ins_tree* (*link* t_1 t_2) *ts*)
)

definition *T_insert* :: *'a::linorder* ⇒ *'a trees* ⇒ *nat* **where**
T_insert x ts = *T_ins_tree* (*Node 0 x* []) *ts* + *1*

lemma *T_ins_tree_simple_bound*: *T_ins_tree t ts* ≤ *length ts* + *1*
by (*induction t ts rule*: *T_ins_tree.induct*) *auto*

48.3.2 *T_insert*

lemma *T_insert_bound*:
assumes *invar ts*
shows *T_insert x ts* ≤ *log 2* (*size* (*mset_trees ts*) + *1*) + *2*
proof −
have *real* (*T_insert x ts*) ≤ *real* (*length ts*) + *2*

unfolding *T_insert_def* **using** *T_ins_tree_simple_bound*
using *of_nat_mono* **by** *fastforce*
also note *size_mset_trees*[*OF* ‹*invar ts*›]
finally show *?thesis* **by** *simp*
qed

48.3.3 *T_merge*

context
includes *pattern_aliases*
begin

fun *T_merge* :: *'a::linorder trees* ⇒ *'a trees* ⇒ *nat* **where**
$T\_merge\ ts_1\ []\ =\ 1$
$|\ T\_merge\ []\ ts_2\ =\ 1$
$|\ T\_merge\ (t_1\#ts_1\ =:\ h_1)\ (t_2\#ts_2\ =:\ h_2)\ =\ 1\ +\ ($
$if\ rank\ t_1\ <\ rank\ t_2\ then\ T\_merge\ ts_1\ h_2$
$else\ if\ rank\ t_2\ <\ rank\ t_1\ then\ T\_merge\ h_1\ ts_2$
$else\ T\_ins\_tree\ (link\ t_1\ t_2)\ (merge\ ts_1\ ts_2)\ +\ T\_merge\ ts_1\ ts_2$
)

end

A crucial idea is to estimate the time in correlation with the result length, as each carry reduces the length of the result.

lemma *T_ins_tree_length*:
T_ins_tree t ts + *length* (*ins_tree t ts*) = *2* + *length ts*
by (*induction t ts rule*: *ins_tree.induct*) *auto*

lemma *T_merge_length*:
$T\_merge\ ts_1\ ts_2\ +\ length\ (merge\ ts_1\ ts_2)\ \leq\ 2\ *\ (length\ ts_1\ +\ length\ ts_2)$
$+\ 1$
by $(induction\ ts_1\ ts_2\ rule:\ T\_merge.induct)$
(*auto simp*: *T_ins_tree_length algebra_simps*)

Finally, we get the desired logarithmic bound

lemma *T_merge_bound*:
fixes $ts_1\ ts_2$
defines $n_1 \equiv size\ (mset\_trees\ ts_1)$
defines $n_2 \equiv size\ (mset\_trees\ ts_2)$
assumes $invar\ ts_1\ invar\ ts_2$
shows $T\_merge\ ts_1\ ts_2 \leq 4*log\ 2\ (n_1 + n_2 + 1) + 1$
proof −
note *n_defs* = *assms*(*1*,*2*)

have *T_merge* ts_1 ts_2 ≤ *2* ∗ *real* (*length* ts_1) + *2* ∗ *real* (*length* ts_2) + *1*
using *T_merge_length*[*of* ts_1 ts_2] **by** *simp*
also note *size_mset_trees*[*OF* ‹*invar* ts_1›]
also note *size_mset_trees*[*OF* ‹*invar* ts_2›]
finally have *T_merge* ts_1 ts_2 ≤ *2* ∗ *log 2* (n_1 + *1*) + *2* ∗ *log 2* (n_2 + *1*) + *1*
unfolding *n_defs* **by** (*simp add*: *algebra_simps*)
also have *log 2* (n_1 + *1*) ≤ *log 2* (n_1 + n_2 + *1*)
unfolding *n_defs* **by** (*simp add*: *algebra_simps*)
also have *log 2* (n_2 + *1*) ≤ *log 2* (n_1 + n_2 + *1*)
unfolding *n_defs* **by** (*simp add*: *algebra_simps*)
finally show *?thesis* **by** (*simp add*: *algebra_simps*)
qed

48.3.4 *T_get_min*

fun *T_get_min* :: *'a*::*linorder trees* ⇒ *nat* **where**
T_get_min [*t*] = *1*
| *T_get_min* (*t*#*ts*) = *1* + *T_get_min ts*

lemma *T_get_min_estimate*: *ts*≠[] ⟹ *T_get_min ts* = *length ts*
by (*induction ts rule*: *T_get_min.induct*) *auto*

lemma *T_get_min_bound*:
assumes *invar ts*
assumes *ts*≠[]
shows *T_get_min ts* ≤ *log 2* (*size* (*mset_trees ts*) + *1*)
proof −
have *1*: *T_get_min ts* = *length ts* **using** *assms T_get_min_estimate* **by** *auto*
also note *size_mset_trees*[*OF* ‹*invar ts*›]
finally show *?thesis* **.**
qed

48.3.5 *T_del_min*

fun *T_get_min_rest* :: *'a*::*linorder trees* ⇒ *nat* **where**
T_get_min_rest [*t*] = *1*
| *T_get_min_rest* (*t*#*ts*) = *1* + *T_get_min_rest ts*

lemma *T_get_min_rest_estimate*: *ts*≠[] ⟹ *T_get_min_rest ts* = *length ts*
by (*induction ts rule*: *T_get_min_rest.induct*) *auto*

lemma *T_get_min_rest_bound*:
assumes *invar ts*
assumes *ts≠[]*
shows *T_get_min_rest ts* ≤ *log 2* (*size* (*mset_trees ts*) + *1*)
proof −
have *1*: *T_get_min_rest ts* = *length ts* **using** *assms T_get_min_rest_estimate* **by** *auto*
also note *size_mset_trees*[*OF* ‹*invar ts*›]
finally show *?thesis* **.**
qed

Note that although the definition of function *rev* has quadratic complexity, it can and is implemented (via suitable code lemmas) as a linear time function. Thus the following definition is justified:

definition *T_rev xs* = *length xs* + *1*

definition *T_del_min* :: *'a::linorder trees* ⇒ *nat* **where**
T_del_min ts = *T_get_min_rest ts* + (*case get_min_rest ts of* (*Node _ x* ts_1, ts_2)
⇒ *T_rev* ts_1 + *T_merge* (*rev* ts_1) ts_2
) + *1*

lemma *T_del_min_bound*:
fixes *ts*
defines *n* ≡ *size* (*mset_trees ts*)
assumes *invar ts* **and** *ts≠[]*
shows *T_del_min ts* ≤ *6* ∗ *log 2* (*n*+*1*) + *3*
proof −
obtain *r x* ts_1 ts_2 **where** *GM*: *get_min_rest ts* = (*Node r x* ts_1, ts_2)
by (*metis surj_pair tree.exhaust_sel*)

have *I1*: *invar* (*rev* ts_1) **and** *I2*: *invar* ts_2
using *invar_get_min_rest*[*OF GM* ‹*ts≠[]*› ‹*invar ts*›] *invar_children*
by *auto*

define n_1 **where** n_1 = *size* (*mset_trees* ts_1)
define n_2 **where** n_2 = *size* (*mset_trees* ts_2)

have $n_1 \leq n$ $n_1 + n_2 \leq n$ **unfolding** *n_def* n_1*_def* n_2*_def*
using *mset_get_min_rest*[*OF GM* ‹*ts≠[]*›]
by (*auto simp*: *mset_trees_def*)

have *T_del_min ts* = *real* (*T_get_min_rest ts*) + *real* (*T_rev* ts_1) +

real (*T_merge* (*rev* ts_1) ts_2) + *1*
    unfolding *T_del_min_def GM*
    by *simp*
  also have *T_get_min_rest ts* ≤ *log 2* (*n*+*1*)
    using *T_get_min_rest_bound*[*OF* ‹*invar ts*› ‹*ts*≠[]›] **unfolding** *n_def*
by *simp*
  also have *T_rev* ts_1 ≤ *1* + *log 2* (n_1 + *1*)
    unfolding *T_rev_def* n_1*_def* **using** *size_mset_trees*[*OF I1*] **by** *simp*
  also have *T_merge* (*rev* ts_1) ts_2 ≤ *4*∗*log 2* (n_1 + n_2 + *1*) + *1*
    unfolding n_1*_def* n_2*_def* **using** *T_merge_bound*[*OF I1 I2*] **by** (*simp*
add: *algebra_simps*)
  finally have *T_del_min ts* ≤ *log 2* (*n*+*1*) + *log 2* (n_1 + *1*) + *4*∗*log 2*
(*real* (n_1 + n_2) + *1*) + *3*
    by (*simp add*: *algebra_simps*)
  also note ‹n_1 + n_2 ≤ *n*›
  also note ‹n_1 ≤ *n*›
  finally show *?thesis* **by** (*simp add*: *algebra_simps*)
qed

end

49 Time functions for various standard library operations

theory *Time_Funs*
  imports *Main*
begin

fun *T_length* :: *'a list* ⇒ *nat* **where**
  T_length [] = *1*
| *T_length* (*x* # *xs*) = *T_length xs* + *1*

lemma *T_length_eq*: *T_length xs* = *length xs* + *1*
  by (*induction xs*) *auto*

lemmas [*simp del*] = *T_length.simps*

fun *T_map* :: (*'a* ⇒ *nat*) ⇒ *'a list* ⇒ *nat* **where**
  T_map T_f [] = *1*
| *T_map T_f* (*x* # *xs*) = *T_f x* + *T_map T_f xs* + *1*

lemma *T_map_eq*: *T_map T_f xs* = (∑ *x*←*xs*. *T_f x*) + *length xs* + *1*

by (*induction xs*) *auto*

lemmas [*simp del*] = *T_map.simps*

fun *T_filter* :: (*'a* ⇒ *nat*) ⇒ *'a list* ⇒ *nat* **where**
T_filter T_p [] = *1*
| *T_filter T_p* (*x* # *xs*) = *T_p x* + *T_filter T_p xs* + *1*

lemma *T_filter_eq*: *T_filter T_p xs* = ($\sum$ *x*←*xs*. *T_p x*) + *length xs* + *1*
by (*induction xs*) *auto*

lemmas [*simp del*] = *T_filter.simps*

fun *T_nth* :: *'a list* ⇒ *nat* ⇒ *nat* **where**
T_nth [] *n* = *1*
| *T_nth* (*x* # *xs*) *n* = (*case n of 0* ⇒ *1* | *Suc n'* ⇒ *T_nth xs n'* + *1*)

lemma *T_nth_eq*: *T_nth xs n* = *min n* (*length xs*) + *1*
by (*induction xs n rule*: *T_nth.induct*) (*auto split*: *nat.splits*)

lemmas [*simp del*] = *T_nth.simps*

fun *T_take* :: *nat* ⇒ *'a list* ⇒ *nat* **where**
T_take n [] = *1*
| *T_take n* (*x* # *xs*) = (*case n of 0* ⇒ *1* | *Suc n'* ⇒ *T_take n' xs* + *1*)

lemma *T_take_eq*: *T_take n xs* = *min n* (*length xs*) + *1*
by (*induction xs arbitrary*: *n*) (*auto split*: *nat.splits*)

fun *T_drop* :: *nat* ⇒ *'a list* ⇒ *nat* **where**
T_drop n [] = *1*
| *T_drop n* (*x* # *xs*) = (*case n of 0* ⇒ *1* | *Suc n'* ⇒ *T_drop n' xs* + *1*)

lemma *T_drop_eq*: *T_drop n xs* = *min n* (*length xs*) + *1*
by (*induction xs arbitrary*: *n*) (*auto split*: *nat.splits*)

end

50 The Median-of-Medians Selection Algorithm

theory *Selection*
imports *Complex_Main Time_Funs Sorting*
begin

Note that there is significant overlap between this theory (which is intended mostly for the Functional Data Structures book) and the Median-of-Medians AFP entry.

50.1 Auxiliary material

lemma *replicate_numeral*: *replicate* (*numeral n*) *x* = *x* # *replicate* (*pred_numeral n*) *x*
by (*simp add*: *numeral_eq_Suc*)

lemma *insort_correct*: *insort xs* = *sort xs*
using *sorted_insort mset_insort* **by** (*metis properties_for_sort*)

lemma *sum_list_replicate* [*simp*]: *sum_list* (*replicate n x*) = *n* ∗ *x*
by (*induction n*) *auto*

lemma *mset_concat*: *mset* (*concat xss*) = *sum_list* (*map mset xss*)
by (*induction xss*) *simp_all*

lemma *set_mset_sum_list* [*simp*]: *set_mset* (*sum_list xs*) = (⋃*x*∈*set xs*. *set_mset x*)
by (*induction xs*) *auto*

lemma *filter_mset_image_mset*:
filter_mset P (*image_mset f A*) = *image_mset f* (*filter_mset* (λ*x*. *P* (*f x*)) *A*)
by (*induction A*) *auto*

lemma *filter_mset_sum_list*: *filter_mset P* (*sum_list xs*) = *sum_list* (*map* (*filter_mset P*) *xs*)
by (*induction xs*) *simp_all*

lemma *sum_mset_mset_mono*:
assumes (⋀*x*. *x* ∈# *A* ⟹ *f x* ⊆# *g x*)
shows (∑*x*∈#*A*. *f x*) ⊆# (∑*x*∈#*A*. *g x*)
using *assms* **by** (*induction A*) (*auto intro*!: *subset_mset.add_mono*)

lemma *mset_filter_mono*:

assumes $A \subseteq\# B \bigwedge x. x \in\# A \Longrightarrow P x \Longrightarrow Q x$
shows *filter_mset* $P A \subseteq\#$ *filter_mset* $Q B$
by (*rule mset_subset_eqI*) (*insert assms*, *auto simp*: *mset_subset_eq_count count_eq_zero_iff*)

lemma *size_mset_sum_mset_distrib*: *size* (*sum_mset* A :: $'a$ *multiset*) = *sum_mset* (*image_mset size* A)
by (*induction* A) *auto*

lemma *sum_mset_mono*:
assumes $\bigwedge x. x \in\# A \Longrightarrow f x \leq (g x :: 'a :: \{ordered\_ab\_semigroup\_add, comm\_monoid\_add\})$
shows $(\sum x{\in}\#A. f x) \leq (\sum x{\in}\#A. g x)$
using *assms* **by** (*induction* A) (*auto intro*!: *add_mono*)

lemma *filter_mset_is_empty_iff*: *filter_mset* $P A = \{\#\} \longleftrightarrow (\forall x. x \in\# A \longrightarrow \neg P x)$
by (*auto simp*: *multiset_eq_iff count_eq_zero_iff*)

lemma *sort_eq_Nil_iff* [*simp*]: *sort xs* = [] $\longleftrightarrow$ *xs* = []
by (*metis set_empty set_sort*)

lemma *sort_mset_cong*: *mset xs* = *mset ys* $\Longrightarrow$ *sort xs* = *sort ys*
by (*metis sorted_list_of_multiset_mset*)

lemma *Min_set_sorted*: *sorted xs* $\Longrightarrow$ *xs* $\neq$ [] $\Longrightarrow$ *Min* (*set xs*) = *hd xs*
by (*cases xs*; *force intro*: *Min_insert2*)

lemma *hd_sort*:
fixes *xs* :: $'a$ :: *linorder list*
shows *xs* $\neq$ [] $\Longrightarrow$ *hd* (*sort xs*) = *Min* (*set xs*)
by (*subst Min_set_sorted* [*symmetric*]) *auto*

lemma *length_filter_conv_size_filter_mset*: *length* (*filter P xs*) = *size* (*filter_mset* P (*mset xs*))
by (*induction xs*) *auto*

lemma *sorted_filter_less_subset_take*:
assumes *sorted xs* **and** $i <$ *length xs*
shows $\{\#x \in\#$ *mset xs*. $x <$ *xs* ! $i\#\} \subseteq\#$ *mset* (*take i xs*)
using *assms*
proof (*induction xs arbitrary*: *i rule*: *list.induct*)
case (*Cons x xs i*)
show *?case*
proof (*cases i*)

case *0*
thus *?thesis* **using** *Cons.prems* **by** (*auto simp*: *filter_mset_is_empty_iff*)
next
case (*Suc i′*)
have {#*y* ∈# *mset* (*x* # *xs*). *y* < (*x* # *xs*) ! *i*#} ⊆# *add_mset x* {#*y* ∈# *mset xs*. *y* < *xs* ! *i′*#}
using *Suc Cons.prems* **by** (*auto*)
also have . . . ⊆# *add_mset x* (*mset* (*take i′ xs*))
unfolding *mset_subset_eq_add_mset_cancel* **using** *Cons.prems Suc*
by (*intro Cons.IH*) (*auto*)
also have . . . = *mset* (*take i* (*x* # *xs*)) **by** (*simp add*: *Suc*)
finally show *?thesis* **.**
qed
qed *auto*

lemma *sorted_filter_greater_subset_drop*:
assumes *sorted xs* **and** *i* < *length xs*
shows {#*x* ∈# *mset xs*. *x* > *xs* ! *i*#} ⊆# *mset* (*drop* (*Suc i*) *xs*)
using *assms*
proof (*induction xs arbitrary*: *i rule*: *list.induct*)
case (*Cons x xs i*)
show *?case*
proof (*cases i*)
case *0*
thus *?thesis* **by** (*auto simp*: *sorted_append filter_mset_is_empty_iff*)
next
case (*Suc i′*)
have {#*y* ∈# *mset* (*x* # *xs*). *y* > (*x* # *xs*) ! *i*#} ⊆# {#*y* ∈# *mset xs*. *y* > *xs* ! *i′*#}
using *Suc Cons.prems* **by** (*auto simp*: *set_conv_nth*)
also have . . . ⊆# *mset* (*drop* (*Suc i′*) *xs*)
using *Cons.prems Suc* **by** (*intro Cons.IH*) (*auto*)
also have . . . = *mset* (*drop* (*Suc i*) (*x* # *xs*)) **by** (*simp add*: *Suc*)
finally show *?thesis* **.**
qed
qed *auto*

50.2 Chopping a list into equally-sized bits

fun *chop* :: *nat* ⇒ *′a list* ⇒ *′a list list* **where**
chop 0 _ = []
| *chop _* [] = []
| *chop n xs* = *take n xs* # *chop n* (*drop n xs*)

lemmas [*simp del*] = *chop.simps*

This is an alternative induction rule for *chop*, which is often nicer to use.

lemma *chop_induct′* [*case_names trivial reduce*]:
  assumes $\bigwedge n\ xs.\ n = 0 \vee xs = [] \Longrightarrow P\ n\ xs$
  assumes $\bigwedge n\ xs.\ n > 0 \Longrightarrow xs \neq [] \Longrightarrow P\ n\ (drop\ n\ xs) \Longrightarrow P\ n\ xs$
  shows $P\ n\ xs$
  using *assms*
proof *induction_schema*
  show *wf* (*measure* (*length* ∘ *snd*))
    by *auto*
qed (*blast* | *simp*)+

lemma *chop_eq_Nil_iff* [*simp*]: $chop\ n\ xs = [] \longleftrightarrow n = 0 \vee xs = []$
  by (*induction n xs rule*: *chop.induct*; *subst chop.simps*) *auto*

lemma *chop_0* [*simp*]: $chop\ 0\ xs = []$
  by (*simp add*: *chop.simps*)

lemma *chop_Nil* [*simp*]: $chop\ n\ [] = []$
  by (*cases n*) (*auto simp*: *chop.simps*)

lemma *chop_reduce*: $n > 0 \Longrightarrow xs \neq [] \Longrightarrow chop\ n\ xs = take\ n\ xs\ \#\ chop\ n\ (drop\ n\ xs)$
  by (*cases n*; *cases xs*) (*auto simp*: *chop.simps*)

lemma *concat_chop* [*simp*]: $n > 0 \Longrightarrow concat\ (chop\ n\ xs) = xs$
  by (*induction n xs rule*: *chop.induct*; *subst chop.simps*) *auto*

lemma *chop_elem_not_Nil* [*dest*]: $ys \in set\ (chop\ n\ xs) \Longrightarrow ys \neq []$
  by (*induction n xs rule*: *chop.induct*; *subst* (*asm*) *chop.simps*)
    (*auto simp*: *eq_commute*[*of* []] *split*: *if_splits*)

lemma *length_chop_part_le*: $ys \in set\ (chop\ n\ xs) \Longrightarrow length\ ys \leq n$
  by (*induction n xs rule*: *chop.induct*; *subst* (*asm*) *chop.simps*) (*auto split*: *if_splits*)

lemma *length_chop*:
  assumes $n > 0$
  shows $length\ (chop\ n\ xs) = nat\ \lceil length\ xs\ /\ n \rceil$
proof −
  from ‹$n > 0$› **have** $real\ n * length\ (chop\ n\ xs) \geq length\ xs$
    by (*induction n xs rule*: *chop.induct*; *subst chop.simps*) (*auto simp*: *field_simps*)

moreover from ‹$n > 0$› **have** *real n* $*$ *length* (*chop n xs*) $<$ *length xs* $+$ *n*
  by (*induction n xs rule*: *chop.induct*; *subst chop.simps*)
    (*auto simp*: *field_simps split*: *nat_diff_split_asm*)+
ultimately have *length* (*chop n xs*) $\geq$ *length xs* / *n* **and** *length* (*chop n xs*) $<$ *length xs* / *n* + *1*
  using *assms* **by** (*auto simp*: *field_simps*)
thus *?thesis* **by** *linarith*
qed

lemma *sum_msets_chop*: $n > 0 \Longrightarrow$ ($\sum$ *ys*←*chop n xs*. *mset ys*) = *mset xs*
  by (*subst mset_concat* [*symmetric*]) *simp_all*

lemma *UN_sets_chop*: $n > 0 \Longrightarrow$ ($\bigcup$ *ys*∈*set* (*chop n xs*). *set ys*) = *set xs*
  by (*simp only*: *set_concat* [*symmetric*] *concat_chop*)

lemma *chop_append*: *d dvd length xs* $\Longrightarrow$ *chop d* (*xs* @ *ys*) = *chop d xs* @ *chop d ys*
  by (*induction d xs rule*: *chop_induct'*) (*auto simp*: *chop_reduce dvd_imp_le*)

lemma *chop_replicate* [*simp*]: $d > 0 \Longrightarrow$ *chop d* (*replicate d xs*) = [*replicate d xs*]
  by (*subst chop_reduce*) *auto*

lemma *chop_replicate_dvd* [*simp*]:
  assumes *d dvd n*
  shows *chop d* (*replicate n x*) = *replicate* (*n div d*) (*replicate d x*)
proof (*cases* $d = 0$)
  case *False*
  from *assms* **obtain** *k* **where** *k*: $n = d * k$
    by *blast*
  have *chop d* (*replicate* ($d * k$) *x*) = *replicate k* (*replicate d x*)
    using *False* **by** (*induction k*) (*auto simp*: *replicate_add chop_append*)
  thus *?thesis* **using** *False* **by** (*simp add*: *k*)
qed *auto*

lemma *chop_concat*:
  assumes ∀ *xs*∈*set xss*. *length xs* = *d* **and** $d > 0$
  shows *chop d* (*concat xss*) = *xss*
  using *assms*
proof (*induction xss*)
  case (*Cons xs xss*)
  have *chop d* (*concat* (*xs* # *xss*)) = *chop d* (*xs* @ *concat xss*)

by *simp*
also have ... = *chop d xs* @ *chop d* (*concat xss*)
using *Cons.prems* **by** (*intro chop_append*) *auto*
also have *chop d xs* = [*xs*]
using *Cons.prems* **by** (*subst chop_reduce*) *auto*
also have *chop d* (*concat xss*) = *xss*
using *Cons.prems* **by** (*intro Cons.IH*) *auto*
finally show *?case* **by** *simp*
qed *auto*

50.3 Selection

definition *select* :: *nat* ⇒ (′*a* :: *linorder*) *list* ⇒ ′*a* **where**
select k xs = *sort xs* ! *k*

lemma *select_0*: *xs* ≠ [] ⟹ *select 0 xs* = *Min* (*set xs*)
by (*simp add*: *hd_sort select_def flip*: *hd_conv_nth*)

lemma *select_mset_cong*: *mset xs* = *mset ys* ⟹ *select k xs* = *select k ys*
using *sort_mset_cong*[*of xs ys*] **unfolding** *select_def* **by** *auto*

lemma *select_in_set* [*intro*,*simp*]:
assumes *k* < *length xs*
shows *select k xs* ∈ *set xs*
proof −
from *assms* **have** *sort xs* ! *k* ∈ *set* (*sort xs*) **by** (*intro nth_mem*) *auto*
also have *set* (*sort xs*) = *set xs* **by** *simp*
finally show *?thesis* **by** (*simp add*: *select_def*)
qed

lemma
assumes *n* < *length xs*
shows *size_less_than_select*: *size* {#*y* ∈# *mset xs*. *y* < *select n xs*#} ≤ *n*
and *size_greater_than_select*: *size* {#*y* ∈# *mset xs*. *y* > *select n xs*#} < *length xs* − *n*
proof −
have *size* {#*y* ∈# *mset* (*sort xs*). *y* < *select n xs*#} ≤ *size* (*mset* (*take n* (*sort xs*)))
unfolding *select_def* **using** *assms*
by (*intro size_mset_mono sorted_filter_less_subset_take*) *auto*
thus *size* {#*y* ∈# *mset xs*. *y* < *select n xs*#} ≤ *n*
by *simp*
have *size* {#*y* ∈# *mset* (*sort xs*). *y* > *select n xs*#} ≤ *size* (*mset* (*drop*

(*Suc n*) (*sort xs*)))
    unfolding *select_def* **using** *assms*
    by (*intro size_mset_mono sorted_filter_greater_subset_drop*) *auto*
  thus *size* {#*y* ∈# *mset xs*. *y* > *select n xs*#} < *length xs* − *n*
    using *assms* **by** *simp*
qed

50.4 The designated median of a list

definition *median* **where** *median xs* = *select* ((*length xs* − *1*) *div 2*) *xs*

lemma *median_in_set* [*intro*, *simp*]:
  assumes *xs* ≠ []
  shows *median xs* ∈ *set xs*
proof −
  from *assms* **have** *length xs* > *0* **by** *auto*
  hence (*length xs* − *1*) *div 2* < *length xs* **by** *linarith*
  thus *?thesis* **by** (*simp add*: *median_def*)
qed

lemma *size_less_than_median*: *size* {#*y* ∈# *mset xs*. *y* < *median xs*#}
≤ (*length xs* − *1*) *div 2*
proof (*cases xs* = [])
  case *False*
  hence *length xs* > *0*
    by *auto*
  hence *less*: (*length xs* − *1*) *div 2* < *length xs*
    by *linarith*
  show *size* {#*y* ∈# *mset xs*. *y* < *median xs*#} ≤ (*length xs* − *1*) *div 2*
    using *size_less_than_select*[*OF less*] **by** (*simp add*: *median_def*)
qed *auto*

lemma *size_greater_than_median*: *size* {#*y* ∈# *mset xs*. *y* > *median xs*#} ≤ *length xs div 2*
proof (*cases xs* = [])
  case *False*
  hence *length xs* > *0*
    by *auto*
  hence *less*: (*length xs* − *1*) *div 2* < *length xs*
    by *linarith*
  have *size* {#*y* ∈# *mset xs*. *y* > *median xs*#} < *length xs* − (*length xs* − *1*) *div 2*
    using *size_greater_than_select*[*OF less*] **by** (*simp add*: *median_def*)
  also have ... = *length xs div 2* + *1*

using ‹*length xs* > *0*› **by** *linarith*
finally show *size* {#*y* ∈# *mset xs*. *y* > *median xs*#} ≤ *length xs div 2*
by *simp*
qed *auto*

lemmas *median_props* = *size_less_than_median size_greater_than_median*

50.5 A recurrence for selection

definition *partition3* :: *'a* ⇒ *'a* :: *linorder list* ⇒ *'a list* × *'a list* × *'a list*
where
partition3 x xs = (*filter* (λ*y*. *y* < *x*) *xs*, *filter* (λ*y*. *y* = *x*) *xs*, *filter* (λ*y*. *y* > *x*) *xs*)

lemma *partition3_code* [*code*]:
partition3 x [] = ([], [], [])
partition3 x (*y* # *ys*) =
(*case partition3 x ys of* (*ls*, *es*, *gs*) ⇒
if y < *x then* (*y* # *ls*, *es*, *gs*) *else if x* = *y then* (*ls*, *y* # *es*, *gs*) *else* (*ls*, *es*, *y* # *gs*))
by (*auto simp*: *partition3_def*)

lemma *sort_append*:
assumes ∀ *x*∈*set xs*. ∀ *y*∈*set ys*. *x* ≤ *y*
shows *sort* (*xs* @ *ys*) = *sort xs* @ *sort ys*
using *assms* **by** (*intro properties_for_sort*) (*auto simp*: *sorted_append*)

lemma *select_append*:
assumes ∀ *y*∈*set ys*. ∀ *z*∈*set zs*. *y* ≤ *z*
shows *k* < *length ys* ⟹ *select k* (*ys* @ *zs*) = *select k ys*
and *k* ∈ {*length ys*..<*length ys* + *length zs*} ⟹
select k (*ys* @ *zs*) = *select* (*k* − *length ys*) *zs*
using *assms* **by** (*simp_all add*: *select_def sort_append nth_append*)

lemma *select_append'*:
assumes ∀ *y*∈*set ys*. ∀ *z*∈*set zs*. *y* ≤ *z* **and** *k* < *length ys* + *length zs*
shows *select k* (*ys* @ *zs*) = (*if k* < *length ys then select k ys else select* (*k* − *length ys*) *zs*)
using *assms* **by** (*auto intro*!: *select_append*)

theorem *select_rec_partition*:
assumes *k* < *length xs*
shows *select k xs* = (
let (*ls*, *es*, *gs*) = *partition3 x xs*

```
        in
          if k < length ls then select k ls
          else if k < length ls + length es then x
          else select (k − length ls − length es) gs
        ) (is _ = ?rhs)
proof −
  define ls es gs where ls = filter (λy. y < x) xs and es = filter (λy. y =
x) xs
                  and gs = filter (λy. y > x) xs
  define nl ne where [simp]: nl = length ls ne = length es
  have mset_eq: mset xs = mset ls + mset es + mset gs
    unfolding ls_def es_def gs_def by (induction xs) auto
  have length_eq: length xs = length ls + length es + length gs
    unfolding ls_def es_def gs_def
    using [[simp_depth_limit = 1]] by (induction xs) auto
  have [simp]: select i es = x if i < length es for i
  proof −
    have select i es ∈ set (sort es) unfolding select_def
      using that by (intro nth_mem) auto
    thus ?thesis
      by (auto simp: es_def)
  qed

  have select k xs = select k (ls @ (es @ gs))
    by (intro select_mset_cong) (simp_all add: mset_eq)
  also have ... = (if k < nl then select k ls else select (k − nl) (es @ gs))
    unfolding nl_ne_def using assms
    by (intro select_append') (auto simp: ls_def es_def gs_def length_eq)
  also have ... = (if k < nl then select k ls else if k < nl + ne then x
                  else select (k − nl − ne) gs)
  proof (rule if_cong)
    assume ¬k < nl
    have select (k − nl) (es @ gs) =
                (if k − nl < ne then select (k − nl) es else select (k − nl −
ne) gs)
      unfolding nl_ne_def using assms ‹¬k < nl›
     by (intro select_append') (auto simp: ls_def es_def gs_def length_eq)
    also have ... = (if k < nl + ne then x else select (k − nl − ne) gs)
      using ‹¬k < nl› by auto
    finally show select (k − nl) (es @ gs) = ... .
  qed simp_all
  also have ... = ?rhs
    by (simp add: partition3_def ls_def es_def gs_def)
  finally show ?thesis .
```

qed

50.6 The size of the lists in the recursive calls

We now derive an upper bound for the number of elements of a list that are smaller (resp. bigger) than the median of medians with chopping size 5. To avoid having to do the same proof twice, we do it generically for an operation $\prec$ that we will later instantiate with either $<$ or $>$.

context
  fixes *xs* :: *'a* :: *linorder list*
  fixes *M* **defines** $M \equiv$ *median* (*map median* (*chop 5 xs*))
begin

lemma *size_median_of_medians_aux*:
  fixes *R* :: *'a* :: *linorder* $\Rightarrow$ *'a* $\Rightarrow$ *bool* (**infix** $\prec$ *50*)
  assumes *R*: $R \in \{(<), (>)\}$
  shows *size* {#*y* ∈# *mset xs*. *y* $\prec$ *M*#} $\leq$ *nat* ⌈*0.7* ∗ *length xs* + *3*⌉
proof –
  define *n* **and** *m* **where** [*simp*]: *n* = *length xs* **and** *m* = *length* (*chop 5 xs*)

We define an abbreviation for the multiset of all the chopped-up groups:

We then split that multiset into those groups whose medians is less than *M* and the rest.

  define *Y_small* ($Y_\prec$) **where** $Y_\prec$ = *filter_mset* (λ*ys*. *median ys* $\prec$ *M*) (*mset* (*chop 5 xs*))
  define *Y_big* ($Y_\succeq$) **where** $Y_\succeq$ = *filter_mset* (λ*ys*. ¬(*median ys* $\prec$ *M*)) (*mset* (*chop 5 xs*))
  have *m* = *size* (*mset* (*chop 5 xs*)) **by** (*simp add*: *m_def*)
  also have *mset* (*chop 5 xs*) = $Y_\prec$ + $Y_\succeq$ **unfolding** *Y_small_def Y_big_def*
    by (*rule multiset_partition*)
  finally have *m_eq*: *m* = *size* $Y_\prec$ + *size* $Y_\succeq$ **by** *simp*

At most half of the lists have a median that is smaller than the median of medians:

  have *size* $Y_\prec$ = *size* (*image_mset median* $Y_\prec$) **by** *simp*
  also have *image_mset median* $Y_\prec$ = {#*y* ∈# *mset* (*map median* (*chop 5 xs*)). *y* $\prec$ *M*#}
   unfolding *Y_small_def* **by** (*subst filter_mset_image_mset* [*symmetric*]) *simp_all*
  also have *size* ... $\leq$ (*length* (*map median* (*chop 5 xs*))) *div 2*
    unfolding *M_def* **using** *median_props*[*of map median* (*chop 5 xs*)] *R* **by** *auto*
  also have ... = *m div 2* **by** (*simp add*: *m_def*)

finally have *size_Y_small*: *size* $Y_{\prec} \leq$ *m div 2* **.**

We estimate the number of elements less than M by grouping them into elements coming from $Y_{\prec}$ and elements coming from $Y_{\succeq}$:

have $\{\#y \in\#$ *mset xs*. $y \prec M\#\} = \{\#y \in\# (\sum ys{\leftarrow}chop\ 5\ xs.\ mset\ ys).$
$y \prec M\#\}$
by (*subst sum_msets_chop*) *simp_all*
also have $\ldots = (\sum ys{\leftarrow}chop\ 5\ xs.\ \{\#y \in\#\ mset\ ys.\ y \prec M\#\})$
by (*subst filter_mset_sum_list*) (*simp add*: *o_def*)
also have $\ldots = (\sum ys{\in}\#mset\ (chop\ 5\ xs).\ \{\#y \in\#\ mset\ ys.\ y \prec M\#\})$
by (*subst sum_mset_sum_list* [*symmetric*]) *simp_all*
also have *mset* (*chop 5 xs*) $= Y_{\prec} + Y_{\succeq}$
by (*simp add*: *Y_small_def Y_big_def not_le*)
also have $(\sum ys{\in}\#\ldots.\ \{\#y \in\#\ mset\ ys.\ y \prec M\#\}) =$
$(\sum ys{\in}\#Y_{\prec}.\ \{\#y \in\#\ mset\ ys.\ y \prec M\#\}) + (\sum ys{\in}\#Y_{\succeq}.$
$\{\#y \in\#\ mset\ ys.\ y \prec M\#\})$
by *simp*

Next, we overapproximate the elements contributed by $Y_{\prec}$: instead of those elements that are smaller than the median, we take *all* the elements of each group. For the elements contributed by $Y_{\succeq}$, we overapproximate by taking all those that are less than their median instead of only those that are less than M.

also have $\ldots \subseteq\# (\sum ys{\in}\#Y_{\prec}.\ mset\ ys) + (\sum ys{\in}\#Y_{\succeq}.\ \{\#y \in\#\ mset$
$ys.\ y \prec median\ ys\#\})$
using R
by (*intro subset_mset.add_mono sum_mset_mset_mono mset_filter_mono*)
(*auto simp*: *Y_big_def*)
finally have *size* $\{\#\ y \in\#\ mset\ xs.\ y \prec M\#\} \leq$ *size* $\ldots$
by (*rule size_mset_mono*)
hence *size* $\{\#\ y \in\#\ mset\ xs.\ y \prec M\#\} \leq$
$(\sum ys{\in}\#Y_{\prec}.\ length\ ys) + (\sum ys{\in}\#Y_{\succeq}.\ size\ \{\#y \in\#\ mset\ ys.\ y$
$\prec median\ ys\#\})$
by (*simp add*: *size_mset_sum_mset_distrib multiset.map_comp o_def*)

Next, we further overapproximate the first sum by noting that each group has at most size 5.

also have $(\sum ys{\in}\#Y_{\prec}.\ length\ ys) \leq (\sum ys{\in}\#Y_{\prec}.\ 5)$
by (*intro sum_mset_mono*) (*auto simp*: *Y_small_def length_chop_part_le*)
also have $\ldots = 5 * size\ Y_{\prec}$ **by** *simp*

Next, we note that each group in $Y_{\succeq}$ can have at most 2 elements that are smaller than its median.

also have $(\sum ys{\in}\#Y_{\succeq}.\ size\ \{\#y \in\#\ mset\ ys.\ y \prec median\ ys\#\}) \leq$

($\sum$ *ys*∈# *Y*$_\succeq$. *length ys div 2*)
proof (*intro sum_mset_mono*, *goal_cases*)
fix *ys* **assume** *ys* ∈# *Y*$_\succeq$
hence *ys* ≠ []
by (*auto simp*: *Y_big_def*)
thus *size* {#*y* ∈# *mset ys*. *y* ≺ *median ys*#} ≤ *length ys div 2*
using *R median_props*[*of ys*] **by** *auto*
qed
also have ... ≤ ($\sum$ *ys*∈# *Y*$_\succeq$. *2*)
by (*intro sum_mset_mono div_le_mono diff_le_mono*)
(*auto simp*: *Y_big_def dest*: *length_chop_part_le*)
also have ... = *2* ∗ *size Y*$_\succeq$ **by** *simp*

Simplifying gives us the main result.

also have *5* ∗ *size Y*$_\prec$ + *2* ∗ *size Y*$_\succeq$ = *2* ∗ *m* + *3* ∗ *size Y*$_\prec$
by (*simp add*: *m_eq*)
also have ... ≤ *3.5* ∗ *m*
using ‹*size Y*$_\prec$ ≤ *m div 2*› **by** *linarith*
also have ... = *3.5* ∗ ⌈*n* / *5*⌉
by (*simp add*: *m_def length_chop*)
also have ... ≤ *0.7* ∗ *n* + *3.5*
by *linarith*
finally have *size* {#*y* ∈# *mset xs*. *y* ≺ *M*#} ≤ *0.7* ∗ *n* + *3.5*
by *simp*
thus *size* {#*y* ∈# *mset xs*. *y* ≺ *M*#} ≤ *nat* ⌈*0.7* ∗ *n* + *3*⌉
by *linarith*
qed

lemma *size_less_than_median_of_medians*:
size {#*y* ∈# *mset xs*. *y* < *M*#} ≤ *nat* ⌈*0.7* ∗ *length xs* + *3*⌉
using *size_median_of_medians_aux*[*of* (<)] **by** *simp*

lemma *size_greater_than_median_of_medians*:
size {#*y* ∈# *mset xs*. *y* > *M*#} ≤ *nat* ⌈*0.7* ∗ *length xs* + *3*⌉
using *size_median_of_medians_aux*[*of* (>)] **by** *simp*

end

50.7 Efficient algorithm

We handle the base cases and computing the median for the chopped-up sublists of size 5 using the naive selection algorithm where we sort the list using insertion sort.

definition *slow_select* **where**

```
  slow_select k xs = insort xs ! k

definition slow_median where
  slow_median xs = slow_select ((length xs - 1) div 2) xs

lemma slow_select_correct: slow_select k xs = select k xs
  by (simp add: slow_select_def select_def insort_correct)

lemma slow_median_correct: slow_median xs = median xs
  by (simp add: median_def slow_median_def slow_select_correct)
```

The definition of the selection algorithm is complicated somewhat by the fact that its termination is contingent on its correctness: if the first recursive call were to return an element for x that is e.g. smaller than all list elements, the algorithm would not terminate.

Therefore, we first prove partial correctness, then termination, and then combine the two to obtain total correctness.

```
function mom_select where
  mom_select k xs = (
    if length xs ≤ 20 then
      slow_select k xs
    else
        let M = mom_select (((length xs + 4) div 5 - 1) div 2) (map
slow_median (chop 5 xs));
          (ls, es, gs) = partition3 M xs
      in
        if k < length ls then mom_select k ls
        else if k < length ls + length es then M
        else mom_select (k - length ls - length es) gs
    )
  by auto
```

If *mom_select* terminates, it agrees with *select*:

```
lemma mom_select_correct_aux:
  assumes mom_select_dom (k, xs) and k < length xs
  shows   mom_select k xs = select k xs
  using assms
proof (induction rule: mom_select.pinduct)
  case (1 k xs)
  show mom_select k xs = select k xs
  proof (cases length xs ≤ 20)
    case True
    thus mom_select k xs = select k xs using 1.prems 1.hyps
     by (subst mom_select.psimps) (auto simp: select_def slow_select_correct)
```

next
  case *False*
  define *x* **where**
   x = mom_select (((length xs + 4) div 5 − 1) div 2) (map slow_median (chop 5 xs))
  define *ls es gs* **where** *ls = filter (λy. y < x) xs* **and** *es = filter (λy. y = x) xs*
    and *gs = filter (λy. y > x) xs*
  define *nl ne* **where** *nl = length ls* **and** *ne = length es*
  note *defs = nl_def ne_def x_def ls_def es_def gs_def*
  have *tw*: *(ls, es, gs) = partition3 x xs*
   unfolding *partition3_def defs One_nat_def* **..**
  have *length_eq*: *length xs = nl + ne + length gs*
   unfolding *nl_def ne_def ls_def es_def gs_def*
   using *[[simp_depth_limit = 1]]* **by** *(induction xs) auto*
  note *IH = 1.IH(2,3)[OF False x_def tw refl refl]*

  have *mom_select k xs = (if k < nl then mom_select k ls else if k < nl + ne then x*
    else mom_select (k − nl − ne) gs) **using** *1.hyps False*
   by *(subst mom_select.psimps) (simp_all add: partition3_def flip: defs One_nat_def)*
  also have *... = (if k < nl then select k ls else if k < nl + ne then x*
    else select (k − nl − ne) gs)
   using *IH length_eq 1.prems* **by** *(simp add: ls_def es_def gs_def nl_def ne_def)*
  also have *... = select k xs* **using** *‹k < length xs›*
   by *(subst (3) select_rec_partition[of _ _ x]) (simp_all add: nl_def ne_def flip: tw)*
  finally show *mom_select k xs = select k xs* **.**
 qed
qed

mom_select indeed terminates for all inputs:

lemma *mom_select_termination*: *All mom_select_dom*
proof *(relation measure (length ∘ snd); (safe)?)*
 fix *k :: nat* **and** *xs :: 'a list*
 assume *¬ length xs ≤ 20*
 thus *((((length xs + 4) div 5 − 1) div 2, map slow_median (chop 5 xs)), k, xs)*
   ∈ measure (length ∘ snd)
  by *(auto simp: length_chop nat_less_iff ceiling_less_iff)*
next

fix *k* :: *nat* **and** *xs ls es gs* :: *'a list*
define *x* **where** *x* = *mom_select* (((*length xs* + *4*) *div 5* − *1*) *div 2*) (*map slow_median* (*chop 5 xs*))
assume *A*: ¬ *length xs* ≤ *20*
(*ls*, *es*, *gs*) = *partition3 x xs*
mom_select_dom (((*length xs* + *4*) *div 5* − *1*) *div 2*,
map slow_median (*chop 5 xs*))
have *less*: ((*length xs* + *4*) *div 5* − *1*) *div 2* < *nat* ⌈*length xs* / *5*⌉
using *A*(*1*) **by** *linarith*

For termination, it suffices to prove that *x* is in the list.

have *x* = *select* (((*length xs* + *4*) *div 5* − *1*) *div 2*) (*map slow_median* (*chop 5 xs*))
using *less* **unfolding** *x_def* **by** (*intro mom_select_correct_aux A*) (*auto simp*: *length_chop*)
also have ... ∈ *set* (*map slow_median* (*chop 5 xs*))
using *less* **by** (*intro select_in_set*) (*simp_all add*: *length_chop*)
also have ... ⊆ *set xs*
unfolding *set_map*
proof *safe*
fix *ys* **assume** *ys*: *ys* ∈ *set* (*chop 5 xs*)
hence *median ys* ∈ *set ys*
by *auto*
also have *set ys* ⊆ ⋃(*set* ' *set* (*chop 5 xs*))
using *ys* **by** *blast*
also have ... = *set xs*
by (*rule UN_sets_chop*) *simp_all*
finally show *slow_median ys* ∈ *set xs*
by (*simp add*: *slow_median_correct*)
qed
finally have *x* ∈ *set xs* **.**
thus ((*k*, *ls*), *k*, *xs*) ∈ *measure* (*length* ∘ *snd*)
and ((*k* − *length ls* − *length es*, *gs*), *k*, *xs*) ∈ *measure* (*length* ∘ *snd*)
using *A*(*1*,*2*) **by** (*auto simp*: *partition3_def intro*!: *length_filter_less*[*of x*])
qed

termination *mom_select* **by** (*rule mom_select_termination*)

lemmas [*simp del*] = *mom_select.simps*

lemma *mom_select_correct*: *k* < *length xs* ⟹ *mom_select k xs* = *select k xs*
using *mom_select_correct_aux* **and** *mom_select_termination* **by** *blast*

50.8 Running time analysis

fun *T_partition3* :: *'a* ⇒ *'a list* ⇒ *nat* **where**
T_partition3 x [] = *1*
| *T_partition3 x* (*y* # *ys*) = *T_partition3 x ys* + *1*

lemma *T_partition3_eq*: *T_partition3 x xs* = *length xs* + *1*
by (*induction x xs rule*: *T_partition3.induct*) *auto*

definition *T_slow_select* :: *nat* ⇒ *'a* :: *linorder list* ⇒ *nat* **where**
T_slow_select k xs = *T_insort xs* + *T_nth* (*insort xs*) *k* + *1*

definition *T_slow_median* :: *'a* :: *linorder list* ⇒ *nat* **where**
T_slow_median xs = *T_slow_select* ((*length xs* − *1*) *div 2*) *xs* + *1*

lemma *T_slow_select_le*: *T_slow_select k xs* ≤ *length xs* ^ *2* + *3* ∗ *length xs* + *3*
proof −
have *T_slow_select k xs* ≤ (*length xs* + *1*)^2 + (*length* (*insort xs*) + *1*) + *1*
unfolding *T_slow_select_def*
by (*intro add_mono T_insort_length*) (*auto simp*: *T_nth_eq*)
also have . . . = *length xs* ^ *2* + *3* ∗ *length xs* + *3*
by (*simp add*: *insort_correct algebra_simps power2_eq_square*)
finally show *?thesis* **.**
qed

lemma *T_slow_median_le*: *T_slow_median xs* ≤ *length xs* ^ *2* + *3* ∗ *length xs* + *4*
unfolding *T_slow_median_def* **using** *T_slow_select_le*[*of* (*length xs* − *1*) *div 2 xs*] **by** *simp*

fun *T_chop* :: *nat* ⇒ *'a list* ⇒ *nat* **where**
T_chop 0 _ = *1*
| *T_chop* _ [] = *1*
| *T_chop n xs* = *T_take n xs* + *T_drop n xs* + *T_chop n* (*drop n xs*)

lemmas [*simp del*] = *T_chop.simps*

lemma *T_chop_Nil* [*simp*]: *T_chop d* [] = *1*
by (*cases d*) (*auto simp*: *T_chop.simps*)

lemma *T_chop_0* [*simp*]: *T_chop 0 xs* = *1* by (*auto simp*: *T_chop.simps*)

lemma *T_chop_reduce*:
n > *0* ⟹ *xs* ≠ [] ⟹ *T_chop n xs* = *T_take n xs* + *T_drop n xs* + *T_chop n* (*drop n xs*)
by (*cases n*; *cases xs*) (*auto simp*: *T_chop.simps*)

lemma *T_chop_le*: *T_chop d xs* ≤ *5* ∗ *length xs* + *1*
by (*induction d xs rule*: *T_chop.induct*) (*auto simp*: *T_chop_reduce T_take_eq T_drop_eq*)

The option *domintros* here allows us to explicitly reason about where the function does and does not terminate. With this, we can skip the termination proof this time because we can reuse the one for *mom_select*.

function (*domintros*) *T_mom_select* :: *nat* ⇒ *'a* :: *linorder list* ⇒ *nat*
where
```
  T_mom_select k xs = (
    if length xs ≤ 20 then
      T_slow_select k xs
    else
      let xss = chop 5 xs;
          ms = map slow_median xss;
          idx = (((length xs + 4) div 5 − 1) div 2);
          x = mom_select idx ms;
          (ls, es, gs) = partition3 x xs;
          nl = length ls;
          ne = length es
      in
        (if k < nl then T_mom_select k ls
         else if k < nl + ne then 0
         else T_mom_select (k − nl − ne) gs) +
        T_mom_select idx ms + T_chop 5 xs + T_map T_slow_median xss +
        T_partition3 x xs + T_length ls + T_length es + 1
     )
```
by *auto*

termination *T_mom_select*
proof (*rule allI*, *safe*)
fix *k* :: *nat* **and** *xs* :: *'a* :: *linorder list*
have *mom_select_dom* (*k*, *xs*)
using *mom_select_termination* **by** *blast*
thus *T_mom_select_dom* (*k*, *xs*)
by (*induction k xs rule*: *mom_select.pinduct*)

  (*rule T_mom_select.domintros*, *simp_all*)
qed

lemmas [*simp del*] = *T_mom_select.simps*

function *T′_mom_select* :: *nat* ⇒ *nat* **where**
 T′_mom_select n =
  (*if n* ≤ *20 then*
   463
  else
   T′_mom_select (*nat* ⌈*0.2*∗*n*⌉) + *T′_mom_select* (*nat* ⌈*0.7*∗*n*+*3*⌉) + *17* ∗ *n* + *50*)
 by *force*+
termination by (*relation measure id*; *simp*; *linarith*)

lemmas [*simp del*] = *T′_mom_select.simps*

lemma *T′_mom_select_ge*: *T′_mom_select n* ≥ *463*
 by (*induction n rule*: *T′_mom_select.induct*; *subst T′_mom_select.simps*) *auto*

lemma *T′_mom_select_mono*:
 m ≤ *n* ⟹ *T′_mom_select m* ≤ *T′_mom_select n*
proof (*induction n arbitrary*: *m rule*: *less_induct*)
 case (*less n m*)
 show *?case*
 proof (*cases m* ≤ *20*)
  case *True*
  hence *T′_mom_select m* = *463*
   by (*subst T′_mom_select.simps*) *auto*
  also have ... ≤ *T′_mom_select n*
   by (*rule T′_mom_select_ge*)
  finally show *?thesis* **.**
 next
  case *False*
  hence *T′_mom_select m* =
     T′_mom_select (*nat* ⌈*0.2*∗*m*⌉) + *T′_mom_select* (*nat* ⌈*0.7*∗*m* + *3*⌉) + *17* ∗ *m* + *50*
   by (*subst T′_mom_select.simps*) *auto*
  also have ... ≤ *T′_mom_select* (*nat* ⌈*0.2*∗*n*⌉) + *T′_mom_select* (*nat* ⌈*0.7*∗*n* + *3*⌉) + *17* ∗ *n* + *50*
   using ‹*m* ≤ *n*› **and** *False* **by** (*intro add_mono less.IH*; *linarith*)

also have ... = *T′_mom_select n*
using ‹*m* ≤ *n*› **and** *False* **by** (*subst T′_mom_select.simps*) *auto*
finally show *?thesis* **.**
qed
qed

lemma *T_mom_select_le_aux*: *T_mom_select k xs* ≤ *T′_mom_select* (*length xs*)
proof (*induction k xs rule*: *T_mom_select.induct*)
case (*1 k xs*)
define *n* **where** [*simp*]: *n* = *length xs*
define *x* **where**
x = *mom_select* (((*length xs* + *4*) *div 5* − *1*) *div 2*) (*map slow_median* (*chop 5 xs*))
define *ls es gs* **where** *ls* = *filter* (λ*y*. *y* < *x*) *xs* **and** *es* = *filter* (λ*y*. *y* = *x*) *xs*
and *gs* = *filter* (λ*y*. *y* > *x*) *xs*
define *nl ne* **where** *nl* = *length ls* **and** *ne* = *length es*
note *defs* = *nl_def ne_def x_def ls_def es_def gs_def*
have *tw*: (*ls*, *es*, *gs*) = *partition3 x xs*
unfolding *partition3_def defs One_nat_def* **..**
note *IH* = *1.IH*(*1*,*2*,*3*)[*OF _ refl refl refl x_def tw refl refl refl refl*]

show *?case*
proof (*cases length xs* ≤ *20*)
case *True* — base case
hence *T_mom_select k xs* ≤ (*length xs*)^2 + *3* ∗ *length xs* + *3*
using *T_slow_select_le*[*of k xs*] **by** (*subst T_mom_select.simps*) *auto*
also have ... ≤ *20*^{2} + *3* ∗ *20* + *3*
using *True* **by** (*intro add_mono power_mono*) *auto*
also have ... ≤ *463*
by *simp*
also have ... = *T′_mom_select* (*length xs*)
using *True* **by** (*simp add*: *T′_mom_select.simps*)
finally show *?thesis* **by** *simp*
next
case *False* — recursive case
have ((*n* + *4*) *div 5* − *1*) *div 2* < *nat* ⌈*n* / *5*⌉
using *False* **unfolding** *n_def* **by** *linarith*
hence *x* = *select* (((*n* + *4*) *div 5* − *1*) *div 2*) (*map slow_median* (*chop 5 xs*))
unfolding *x_def n_def* **by** (*intro mom_select_correct*) (*auto simp*: *length_chop*)
also have ((*n* + *4*) *div 5* − *1*) *div 2* = (*nat* ⌈*n* / *5*⌉ − *1*) *div 2*

by *linarith*
also have *select* ... (*map slow_median* (*chop 5 xs*)) = *median* (*map slow_median* (*chop 5 xs*))
by (*auto simp*: *median_def length_chop*)
finally have *x_eq*: x = *median* (*map slow_median* (*chop 5 xs*)) .

The cost of computing the medians of all the subgroups:

define *T_ms* **where** *T_ms* = *T_map T_slow_median* (*chop 5 xs*)
have *T_ms* $\leq$ *9* $*$ *n* + *45*
proof −
have *T_ms* = ($\sum$ *ys*$\leftarrow$*chop 5 xs*. *T_slow_median ys*) + *length* (*chop 5 xs*) + *1*
by (*simp add*: *T_ms_def T_map_eq*)
also have ($\sum$ *ys*$\leftarrow$*chop 5 xs*. *T_slow_median ys*) $\leq$ ($\sum$ *ys*$\leftarrow$*chop 5 xs*. *44*)
proof (*intro sum_list_mono*)
fix *ys* **assume** *ys* $\in$ *set* (*chop 5 xs*)
hence *length ys* $\leq$ *5*
using *length_chop_part_le* **by** *blast*
have *T_slow_median ys* $\leq$ (*length ys*) ^ *2* + *3* $*$ *length ys* + *4*
by (*rule T_slow_median_le*)
also have ... $\leq$ *5* ^ *2* + *3* $*$ *5* + *4*
using ‹*length ys* $\leq$ *5*› **by** (*intro add_mono power_mono*) *auto*
finally show *T_slow_median ys* $\leq$ *44* **by** *simp*
qed
also have ($\sum$ *ys*$\leftarrow$*chop 5 xs*. *44*) + *length* (*chop 5 xs*) + *1* =
45 $*$ *nat* $\lceil$*real n* / *5*$\rceil$ + *1*
by (*simp add*: *map_replicate_const length_chop*)
also have ... $\leq$ *9* $*$ *n* + *45*
by *linarith*
finally show *T_ms* $\leq$ *9* $*$ *n* + *45* **by** *simp*
qed

The cost of the first recursive call (to compute the median of medians):

define *T_rec1* **where**
T_rec1 = *T_mom_select* (((*length xs* + *4*) *div 5* − *1*) *div 2*) (*map slow_median* (*chop 5 xs*))
have *T_rec1* $\leq$ *T'_mom_select* (*length* (*map slow_median* (*chop 5 xs*)))
using *False* **unfolding** *T_rec1_def* **by** (*intro IH*(*3*)) *auto*
hence *T_rec1* $\leq$ *T'_mom_select* (*nat* $\lceil$*0.2* $*$ *n*$\rceil$)
by (*simp add*: *length_chop*)

The cost of the second recursive call (to compute the final result):

define *T_rec2* **where** *T_rec2* = (*if k* < *nl then T_mom_select k ls*

else if k < nl + ne then 0
else T_mom_select (k − nl − ne) gs)
consider *k < nl* | *k ∈ {nl..<nl+ne}* | *k ≥ nl+ne*
by *force*
hence *T_rec2 ≤ T′_mom_select* (*nat* ⌈*0.7 ∗ n + 3*⌉)
proof *cases*
assume *k < nl*
hence *T_rec2 = T_mom_select k ls*
by (*simp add*: *T_rec2_def*)
also have . . . ≤ *T′_mom_select* (*length ls*)
by (*rule IH*(*1*)) (*use* ‹*k < nl*› *False* **in** ‹*auto simp*: *defs*›)
also have *length ls ≤ nat* ⌈*0.7 ∗ n + 3*⌉
unfolding *ls_def* **using** *size_less_than_median_of_medians*[*of xs*]
by (*auto simp*: *length_filter_conv_size_filter_mset slow_median_correct*[*abs_def*] *x_eq*)
hence *T′_mom_select* (*length ls*) ≤ *T′_mom_select* (*nat* ⌈*0.7 ∗ n* + *3*⌉)
by (*rule T′_mom_select_mono*)
finally show *?thesis* **.**
next
assume *k ∈ {nl..<nl + ne}*
hence *T_rec2 = 0*
by (*simp add*: *T_rec2_def*)
thus *?thesis*
using *T′_mom_select_ge*[*of nat* ⌈*0.7 ∗ n + 3*⌉] **by** *simp*
next
assume *k ≥ nl + ne*
hence *T_rec2 = T_mom_select* (*k − nl − ne*) *gs*
by (*simp add*: *T_rec2_def*)
also have . . . ≤ *T′_mom_select* (*length gs*)
unfolding *nl_def ne_def* **by** (*rule IH*(*2*)) (*use* ‹*k ≥ nl + ne*› *False* **in** ‹*auto simp*: *defs*›)
also have *length gs ≤ nat* ⌈*0.7 ∗ n + 3*⌉
unfolding *gs_def* **using** *size_greater_than_median_of_medians*[*of xs*]
by (*auto simp*: *length_filter_conv_size_filter_mset slow_median_correct*[*abs_def*] *x_eq*)
hence *T′_mom_select* (*length gs*) ≤ *T′_mom_select* (*nat* ⌈*0.7 ∗ n* + *3*⌉)
by (*rule T′_mom_select_mono*)
finally show *?thesis* **.**
qed

Now for the final inequality chain:

have *T_mom_select k xs* = *T_rec2* + *T_rec1* + *T_ms* + *n* + *nl* + *ne* + *T_chop 5 xs* + *4* **using** *False*
by (*subst T_mom_select.simps*, *unfold Let_def tw* [*symmetric*] *defs* [*symmetric*])
(*simp_all add*: *nl_def ne_def T_rec1_def T_rec2_def T_partition3_eq T_length_eq T_ms_def*)
also have *nl* ≤ *n* **by** (*simp add*: *nl_def ls_def*)
also have *ne* ≤ *n* **by** (*simp add*: *ne_def es_def*)
also note ‹*T_ms* ≤ *9* ∗ *n* + *45*›
also have *T_chop 5 xs* ≤ *5* ∗ *n* + *1*
using *T_chop_le*[*of 5 xs*] **by** *simp*
also note ‹*T_rec1* ≤ *T′_mom_select* (*nat* ⌈*0.2*∗*n*⌉)›
also note ‹*T_rec2* ≤ *T′_mom_select* (*nat* ⌈*0.7*∗*n* + *3*⌉)›
finally have *T_mom_select k xs* ≤
T′_mom_select (*nat* ⌈*0.7*∗*n* + *3*⌉) + *T′_mom_select* (*nat* ⌈*0.2*∗*n*⌉) + *17* ∗ *n* + *50*
by *simp*
also have . . . = *T′_mom_select n*
using *False* **by** (*subst T′_mom_select.simps*) *auto*
finally show *?thesis* **by** *simp*
qed
qed

50.9 Akra–Bazzi Light

lemma *akra_bazzi_light_aux1*:
fixes *a b* :: *real* **and** *n n0* :: *nat*
assumes *ab*: *a* > *0 a* < *1 n* > *n0*
assumes *n0* ≥ (*max 0 b* + *1*) / (*1* − *a*)
shows *nat* ⌈*a*∗*n*+*b*⌉ < *n*
proof −
have *a* ∗ *real n* + *max 0 b* ≥ *0*
using *ab* **by** *simp*
hence *real* (*nat* ⌈*a*∗*n*+*b*⌉) ≤ *a* ∗ *n* + *max 0 b* + *1*
by *linarith*
also {
have *n0* ≥ (*max 0 b* + *1*) / (*1* − *a*)
by *fact*
also have . . . < *real n*
using *assms* **by** *simp*
finally have *a* ∗ *real n* + *max 0 b* + *1* < *real n*
using *ab* **by** (*simp add*: *field_simps*)
}
finally show *nat* ⌈*a*∗*n*+*b*⌉ < *n*

```
    using ‹n > n0› by linarith
qed

lemma akra_bazzi_light_aux2:
  fixes f :: nat ⇒ real
  fixes n_0 :: nat and a b c d :: real and C1 C2 C_1 C_2 :: real
  assumes bounds: a > 0 c > 0 a + c < 1 C_1 ≥ 0
  assumes rec: ∀ n>n_0. f n = f (nat ⌈a∗n+b⌉) + f (nat ⌈c∗n+d⌉) + C_1 ∗
n + C_2
  assumes ineqs: n_0 > (max 0 b + max 0 d + 2) / (1 − a − c)
               C_3 ≥ C_1 / (1 − a − c)
               C_3 ≥ (C_1 ∗ n_0 + C_2 + C_4) / ((1 − a − c) ∗ n_0 − max 0 b
− max 0 d − 2)
               ∀ n≤n_0. f n ≤ C_4
  shows   f n ≤ C_3 ∗ n + C_4
proof (induction n rule: less_induct)
  case (less n)
  have 0 ≤ C_1 / (1 − a − c)
    using bounds by auto
  also have ... ≤ C_3
    by fact
  finally have C_3 ≥ 0 .

  show ?case
  proof (cases n > n_0)
    case False
    hence f n ≤ C_4
      using ineqs(4) by auto
    also have ... ≤ C_3 ∗ real n + C_4
      using bounds ‹C_3 ≥ 0› by auto
    finally show ?thesis .
  next
    case True
    have nonneg: a ∗ n ≥ 0 c ∗ n ≥ 0
      using bounds by simp_all

    have (max 0 b + 1) / (1 − a) ≤ (max 0 b + max 0 d + 2) / (1 − a
− c)
      using bounds by (intro frac_le) auto
    hence n_0 ≥ (max 0 b + 1) / (1 − a)
      using ineqs(1) by linarith
    hence rec_less1: nat ⌈a∗n+b⌉ < n
     using bounds ‹n > n_0› by (intro akra_bazzi_light_aux1[of _ n_0]) auto
```

have (*max 0 d* + *1*) / (*1* − *c*) ≤ (*max 0 b* + *max 0 d* + *2*) / (*1* − *a* − *c*)
using *bounds* **by** (*intro frac_le*) *auto*
hence n_0 ≥ (*max 0 d* + *1*) / (*1* − *c*)
using *ineqs*(*1*) **by** *linarith*
hence *rec_less2*: *nat* ⌈*c*∗*n*+*d*⌉ < *n*
using *bounds* ‹*n* > n_0› **by** (*intro akra_bazzi_light_aux1*[*of* _ n_0]) *auto*

have *f n* = *f* (*nat* ⌈*a*∗*n*+*b*⌉) + *f* (*nat* ⌈*c*∗*n*+*d*⌉) + C_1 ∗ *n* + C_2
using ‹*n* > n_0› **by** (*subst rec*) *auto*
also have ... ≤ (C_3 ∗ *nat* ⌈*a*∗*n*+*b*⌉ + C_4) + (C_3 ∗ *nat* ⌈*c*∗*n*+*d*⌉ + C_4) + C_1 ∗ *n* + C_2
using *rec_less1 rec_less2* **by** (*intro add_mono less.IH*) *auto*
also have ... ≤ (C_3 ∗ (*a*∗*n*+*max 0 b*+*1*) + C_4) + (C_3 ∗ (*c*∗*n*+*max 0 d*+*1*) + C_4) + C_1 ∗ *n* + C_2
using *bounds* ‹C_3 ≥ *0*› *nonneg* **by** (*intro add_mono mult_left_mono order.refl*; *linarith*)
also have ... = C_3 ∗ *n* + ((C_3 ∗ (*max 0 b* + *max 0 d* + *2*) + *2* ∗ C_4 + C_2) −
(C_3 ∗ (*1* − *a* − *c*) − C_1) ∗ *n*)
by (*simp add*: *algebra_simps*)
also have ... ≤ C_3 ∗ *n* + ((C_3 ∗ (*max 0 b* + *max 0 d* + *2*) + *2* ∗ C_4 + C_2) −
(C_3 ∗ (*1* − *a* − *c*) − C_1) ∗ n_0)
using ‹*n* > n_0› *ineqs*(*2*) *bounds*
by (*intro add_mono diff_mono order.refl mult_left_mono*) (*auto simp*: *field_simps*)
also have (C_3 ∗ (*max 0 b* + *max 0 d* + *2*) + *2* ∗ C_4 + C_2) − (C_3 ∗ (*1* − *a* − *c*) − C_1) ∗ n_0 ≤ C_4
using *ineqs bounds* **by** (*simp add*: *field_simps*)
finally show *f n* ≤ C_3 ∗ *real n* + C_4
by (*simp add*: *mult_right_mono*)
qed
qed

lemma *akra_bazzi_light*:
fixes *f* :: *nat* ⇒ *real*
fixes n_0 :: *nat* **and** *a b c d* C_1 C_2 :: *real*
assumes *bounds*: *a* > *0* *c* > *0* *a* + *c* < *1* C_1 ≥ *0*
assumes *rec*: ∀ *n*>n_0. *f n* = *f* (*nat* ⌈*a*∗*n*+*b*⌉) + *f* (*nat* ⌈*c*∗*n*+*d*⌉) + C_1 ∗ *n* + C_2
shows ∃ C_3 C_4. ∀ *n*. *f n* ≤ C_3 ∗ *real n* + C_4
proof −
define n_0' **where** n_0' = *max* n_0 (*nat* ⌈(*max 0 b* + *max 0 d* + *2*) / (*1* −

$a - c) + 1\rceil)$
  define C_4 **where** $C_4 = Max\ (f\ `\ \{..n_0'\})$
  define C_3 **where** $C_3 = max\ (C_1\ /\ (1 - a - c))$
  $((C_1 * n_0' + C_2 + C_4)\ /\ ((1 - a - c) * n_0' - max\ 0\ b - max\ 0\ d - 2))$

  have $f\ n \leq C_3 * n + C_4$ **for** n
  proof (*rule akra_bazzi_light_aux2*[*OF bounds _*])
    show $\forall n>n_0'.\ f\ n = f\ (nat\ \lceil a*n+b\rceil) + f\ (nat\ \lceil c*n+d\rceil) + C_1 * n + C_2$
      using *rec* **by** (*auto simp*: n_0'*_def*)
  next
    show $C_3 \geq C_1\ /\ (1 - a - c)$
     and $C_3 \geq (C_1 * n_0' + C_2 + C_4)\ /\ ((1 - a - c) * n_0' - max\ 0\ b - max\ 0\ d - 2)$
      by (*simp_all add*: C_3*_def*)
  next
    have $(max\ 0\ b + max\ 0\ d + 2)\ /\ (1 - a - c) < nat\ \lceil(max\ 0\ b + max\ 0\ d + 2)\ /\ (1 - a - c) + 1\rceil$
      by *linarith*
    also have $\ldots \leq n_0'$
      by (*simp add*: n_0'*_def*)
    finally show $(max\ 0\ b + max\ 0\ d + 2)\ /\ (1 - a - c) < real\ n_0'$ **.**
  next
    show $\forall n\leq n_0'.\ f\ n \leq C_4$
      by (*auto simp*: C_4*_def*)
  qed
  thus *?thesis* **by** *blast*
qed

lemma *akra_bazzi_light_nat*:
  fixes $f :: nat \Rightarrow nat$
  fixes $n_0 :: nat$ **and** $a\ b\ c\ d :: real$ **and** $C_1\ C_2 :: nat$
  assumes *bounds*: $a > 0\ c > 0\ a + c < 1\ C_1 \geq 0$
  assumes *rec*: $\forall n>n_0.\ f\ n = f\ (nat\ \lceil a*n+b\rceil) + f\ (nat\ \lceil c*n+d\rceil) + C_1 * n + C_2$
  shows $\exists C_3\ C_4.\ \forall n.\ f\ n \leq C_3 * n + C_4$
proof −
  have $\exists C_3\ C_4.\ \forall n.\ real\ (f\ n) \leq C_3 * real\ n + C_4$
    using *assms* **by** (*intro akra_bazzi_light*[*of* $a\ c\ C_1\ n_0\ f\ b\ d\ C_2$]) *auto*
  then obtain $C_3\ C_4$ **where** *le*: $\forall n.\ real\ (f\ n) \leq C_3 * real\ n + C_4$
    by *blast*
  have $f\ n \leq nat\ \lceil C_3\rceil * n + nat\ \lceil C_4\rceil$ **for** n
  proof −

have *real* $(f\ n) \le C_3 * real\ n + C_4$
using *le* **by** *blast*
also have $\ldots \le real\ (nat\ \lceil C_3 \rceil) * real\ n + real\ (nat\ \lceil C_4 \rceil)$
by (*intro add_mono mult_right_mono*; *linarith*)
also have $\ldots = real\ (nat\ \lceil C_3 \rceil * n + nat\ \lceil C_4 \rceil)$
by *simp*
finally show *?thesis* **by** *linarith*
qed
thus *?thesis* **by** *blast*
qed

lemma *T'_mom_select_le'*: $\exists\, C_1\ C_2.\ \forall\, n.\ T'\_mom\_select\ n \le C_1 * n + C_2$
proof (*rule akra_bazzi_light_nat*)
show $\forall\, n{>}20.\ T'\_mom\_select\ n = T'\_mom\_select\ (nat\ \lceil 0.2 * n + 0 \rceil) +$
$T'\_mom\_select\ (nat\ \lceil 0.7 * n + 3 \rceil) + 17 * n + 50$
using *T'_mom_select.simps* **by** *auto*
qed *auto*

end

51 Bibliographic Notes

Red-black trees The insert function follows Okasaki [15]. The delete function in theory *RBT_Set* follows Kahrs [11, 12], an alternative delete function is given in theory *RBT_Set2*.

2-3 trees Equational definitions were given by Hoffmann and O'Donnell [9] (only insertion) and Reade [19]. Our formalisation is based on the teaching material by Turbak [22] and the article by Hinze [8].

1-2 brother trees They were invented by Ottmann and Six [16, 17]. The functional version is due to Hinze [7].

AA trees They were invented by Arne Anderson [3]. Our formalisation follows Ragde [18] but fixes a number of mistakes.

Splay trees They were invented by Sleator and Tarjan [21]. Our formalisation follows Schoenmakers [20].

Join-based BSTs They were invented by Adams [1, 2] and analyzed by Blelloch *et al.* [4].

Leftist heaps They were invented by Crane [6]. A first functional implementation is due to Núñez *et al.* [14].

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